M421, Introduction to Topology I
Assignment 9

Drew Robertson

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Theorem 1. *Let* $(X, \mathscr{T})$ *be a space and* $A \subseteq X$*, then* $A \in \mathscr{T} \Leftrightarrow \mathscr{T}_A \subseteq \mathscr{T}$.

Proof. ($\Rightarrow$) Suppose $A \in \mathscr{T}$, and let $U \in \mathscr{T}_A$. Then because $\mathscr{T}_A = \{V \cap A : V \in \mathscr{T}\}$ and $U \in \mathscr{T}_A$, $\exists V \in \mathscr{T} \ni: U = V \cap A$. And since $A, V \in \mathscr{T}$, $U \in \mathscr{T}$.
$\therefore A \in \mathscr{T} \Rightarrow \mathscr{T}_A \subseteq \mathscr{T}$. □

Proof. ($\Leftarrow$) Suppose $\mathscr{T}_A \subseteq \mathscr{T}$. Then because $\mathscr{T}_A = \{V \cap A : V \in \mathscr{T}\}$, $\forall U \in \mathscr{T}_A, U \in \mathscr{T}$. Since $A \in \mathscr{T}_A$, $A \in \mathscr{T}$.
$\therefore \mathscr{T}_A \subseteq \mathscr{T} \Rightarrow A \in \mathscr{T}$. □

Theorem 2. *Let* $(X, \mathscr{T})$, $(Y, \mathscr{S})$ *be spaces, and* $f : X \to Y$ *be a continuous function. Then* $\forall$ *closed subsets* F *of* Y $f^{-1}(F)$ *is closed in* X.

Proof. Since F is closed $Y \backslash F \in \mathscr{S}$. Since f is continuous, $f^{-1}(Y \backslash F) \in \mathscr{T}$. Note now that $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F) \in \mathscr{T}$.
$\therefore f^{-1}(F)$ is closed in X. □

Theorem 3. *Let* $(X, \mathscr{T})$, $(Y, \mathscr{S})$, $(Z, \mathscr{F})$ *be spaces, and if* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then* $(g \circ f) : X \to Z$ *is continuous.*

Proof. Note that $(g \circ f)^{-1}(V) = f^{-1}\left(g^{-1}(V)\right)$. Let $V \in \mathscr{F}$. Since g is continuous, $g^{-1}(V) \in \mathscr{S}$. Now since f is continuous

$$f^{-1}\left(g^{-1}(V)\right) \in \mathscr{T}, \qquad \text{That is}$$
$$(g \circ f)^{-1}(V) \in \mathscr{T}$$

□