M421, Introduction to Topology I
Assignment 8

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Theorem 1. *Let $(X, \mathscr{T})$ be a space, and let $A \subseteq X$, then Int$(A) = \bigcup\{U \subseteq X : U \in \mathscr{T}$ and $U \subseteq A\}$.*

Proof. Let $x \in X$.

Proof. $(\subseteq)$
Let $x \in \text{Int}(A)$, then by definition $\exists U \in \mathscr{T}$ s.t. $x \in U \subseteq A$. Thus $x \in \bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\}$. □

Proof. $(\supseteq)$ Let $x \in \bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\}$, then $\exists$ a $U_\alpha \in \bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\}$ s.t. $x \in U_\alpha \subset A$. Thus by definition, we have x in a $U \in \mathscr{T}$ s.t. $U \subseteq A$, thus $x \in \text{Int}(A)$. □

□

Theorem 2. *Let $(X, \mathscr{T})$ be a space, and let $A \subseteq X$, then Int(A) is an open set (ie Int$(A) \in \mathscr{T}$).*

Proof. By previous theorem,

$$\text{Int}(A) = \bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\},$$

thus $\text{Int}(A)$ is the union of open sets, and by a previous theorem, the union of open sets is open, thus $\text{Int}(A) \in \mathscr{T}$. □

Theorem 3. *Let $(X, \mathscr{T})$ be a space, and let $A \subseteq X$. Then $A \in \mathscr{T} \Leftrightarrow A =$ Int(A).*

Proof. Proof. $(\Rightarrow)$
Let $A \in \mathscr{T}$. We wish to show that $A = \text{Int}(A)$.
Suppose $A \neq \text{Int}(A)$. Then $A \neq \bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\}$. Then either

$$(1) \qquad A \not\subseteq \bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\}, \quad \text{or}$$
$$(2) \qquad \bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\} \not\subseteq A$$

However, because $A \in \mathscr{T}$, $A \subseteq \bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\}$. So if $A \neq \text{Int}(A)$, then

$$\bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\} \not\subseteq A.$$

However, it is clear that $\bigcup\{U \subseteq X : U \in \mathscr{T} \text{ and } U \subseteq A\} \subseteq A$, since $U \subseteq A \forall U$. □

Proof. $(\Leftarrow)$
Suppose $A = \text{Int}(A)$. By the previous theorem, $\text{Int}(A) \in \mathscr{T}$, thus $A \in \mathscr{T}$. □

□