M421, Introduction to Topology I
Assignment 5

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Section 2.2 number 11.
Determine if the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2 & \text{if } x \geq 1 \\ -2 & \text{if } x < 1 \end{cases}$$

is:
(a) $\mathscr{U} - \mathscr{U}$ continuous
(b) $\mathscr{U} - \mathscr{H}$ continuous
(c) $\mathscr{U} - \mathscr{C}$ continuous
(d) $\mathscr{H} - \mathscr{U}$ continuous
(e) $\mathscr{U} - \mathscr{H}$ continuous
(f) $\mathscr{C} - \mathscr{H}$ continuous
(g) $\mathscr{C} - \mathscr{C}$ continuous.

Solution
(a) $f^{-1}((1,3)) = [2,\infty) \Rightarrow$ not $\mathscr{U} - \mathscr{U}$ continuous.
(b) Since $\mathscr{U} \subset \mathscr{H}$, by (a) f is not $\mathscr{U} - \mathscr{H}$ continuous.
(c) $f^{-1}((1,\infty)) = [2,\infty) \Rightarrow$ not $\mathscr{U} - \mathscr{C}$ continuous.
(d) Let $V \subseteq \mathbb{R}$, then

$$f^{-1}(V) = \begin{cases} \mathbb{R} & \text{if } 2,-2 \in V \\ [1,\infty) & \text{if } 2 \in V, -2 \notin V \\ (-\infty,1) & \text{if } -2 \in V, 2 \notin V \\ \emptyset & \text{if } -2, 2 \notin V \end{cases}$$

$\mathbb{R}, \emptyset \in \mathscr{H}$ by definition. If $x \in [1,\infty)$, then $\exists b \in [1,\infty)$ s.t. $x \in [1,b) \subset [1,\infty) \Rightarrow \mathscr{H}-$ open. If $x \in (-\infty,1)$, then $\exists a \in (-\infty,1)$ s.t. $x \in [a,1) \subset (-\infty,1)$.
(e) Because $\mathscr{U} \subset \mathscr{H}$, (d)$\Rightarrow$ (e).
(f) $f^{-1}([1,3)) = [2,\infty) \not\subseteq \mathscr{C}$, therefore not $\mathscr{C} - \mathscr{H}$ continuous.
(g) $f^{-1}((1,\infty)) = [2,\infty) \not\subseteq \mathscr{C}$, therefore no $\mathscr{C} - \mathscr{C}$ continuous.

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Let $X = \{a,b,c\}$ and $\mathscr{T} = \{X, \emptyset, \{b\}, \{a,b\}, \{b,c\}$.
(a)

$$\text{Closed Sets } = \emptyset, X, \{a,c\}, \{c\}, \{a\}$$

(b)

$$\text{Cl}(\{b\}) = X$$

(c)

$$\text{Cl}(\{a\}) = \{a\}$$

(d)

$$\text{Cl}(\{a, b\}) = X$$

(e)

$$\text{Cl}(\{b\}) = X,$$

and by definition, if the closure of a set is X, then the set is dense, and since $\{b\}$ is also a proper subset of X.

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Show that $\{1, 2\}$ is a $\mathscr{U}-$ closed subset of $\mathbb{R}$.

Proof. This is equivalent to showing that $\mathbb{R}\backslash\{1, 2\}$ is $\mathscr{U}-$open. Note that

$$\mathbb{R}\backslash\{1, 2\} = (-\infty, 1) \cup (1, 2) \cup (2, \infty),$$

and since this a union of open sets, and because the union of open sets is open, $\mathbb{R}\backslash\{1, 2\}$ is open, and hence $\{1, 2\}$ is closed. □