M421, Introduction to Topology I
Assignment 5

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Theorem 1. *Let X be a non-empty set. Let $x \in X$. Then*

$$\mathscr{T} = \{U \subseteq X : U = \emptyset \text{ or } x \in U\}$$

is a topology.

Proof. Recall that in order to be a topology three conditions must be met:
1. $X, \emptyset \in \mathscr{T}$,

2. $\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathscr{T}$,

3. $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$.

Proof. (1)
$\emptyset \in \mathscr{T}$ by definition. Since $x \in X$, $X \in \mathscr{T}$. □

Proof. (2)
Let $U_\alpha \in \mathscr{T} \forall \alpha \in \Lambda$. We wish to show that $\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathscr{T}$. If $\bigcup_\alpha U_\alpha = \emptyset$, then $\bigcup_\alpha U_\alpha \in \mathscr{T}$. If $\bigcup_\alpha U_\alpha \neq \emptyset, \exists \beta \in \Lambda$ s.t. U_β is non-empty. Since $U_\beta \in \mathscr{T}$, $x \in U_\beta \Rightarrow x \in \bigcup_\alpha U_\alpha \Rightarrow \bigcup_\alpha U_\alpha \in \mathscr{T}$. □

Proof. (3)
Let $U_i \in \mathscr{T} \forall i \in \{1, 2, ..., n\}$. We wish to show that $\bigcap_{i=1}^n U_i \in \mathscr{T}$. If $\bigcap_{i=1}^n U_i = \emptyset$, then $\bigcap_{i=1}^n U_i \in \mathscr{T}$. If $\bigcap_{i=1}^n U_i \neq \emptyset$, then $\forall i \in \{1, 2, ..., n\}$,$U_i$ is non-empty. Since $U_i \in \mathscr{T} \forall i$, $x \in U_i \forall i$. Thus $x \in \bigcap_{i=1}^n U_i \Rightarrow \bigcap_{i=1}^n U_i \in \mathscr{T}$. □

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