M360
Elements of Probability; Assignment 5

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Section 2.2, # 3,12 & Section 2.3, # 4,7a

Question 1. *Find $E(X)$ for each of the distributions given in exercise 2.1-3.*

Solution: Recall, the distributions from exercise 2.1-3 are

$$\text{(a)} \qquad f(x) = \frac{x}{10}, \qquad x \in \{1, 2, 3, 4\}$$

$$\text{(b)} \qquad f(x) = \frac{x}{55}, \qquad x \in \{1, 2, ..., 9, 10\}$$

$$\text{(c)} \qquad f(x) = 3\left(\frac{1}{4}\right)^x, \qquad x \in \mathbb{N}$$

$$\text{(d)} \qquad f(x) = 30\,(x+1)^2\,, \qquad x \in \{0, 1, 2, 3\}$$

$$\text{(e)} \qquad f(x) = \frac{2x}{n(n+1)}, \qquad x \in \{1, 2, ..., n-1, n\}$$

So we will make use of the fact here that $u(x) = x$ for all parts of this problem, and thuse the mathematical expectation is

$$E(X) = \sum_{x \in S} x f(x) \tag{1}$$

So for (a), $E(X) = \sum_{x \in S} x(x/10) = 3$, for (b), $E(X) = \sum_{x \in S} x(x/55) = 7$. For (c), $E(X) = 3\sum_{x \in S} x(1/4)^x = 4/3$, for (d) $E(X) = 30\sum_{x \in S} x(x+1)^2 = 2100$, and for (e), $E(X) = \sum_{x \in S} (2x/\,(n(n+1))) = 2n + 1/3$.

□

Question 2. *In the casino game High-Low, there are three possible bets. Assume that \$ 1 is the size of the bet. A pair if fair-sided dice is rolled. If you bet low, you win \$ 1 if the sum is* $\{2, 3, 4, 5, 6\}$*. If you bet High, you win \$ 1, if the sum of the dice is* $\{8, 9, 10, 11, 12\}$*. If you bet on* $\{7\}$*, you win \$ 4 if a sum of 7 is rolled. In all three cases, your original dollar is returned if you win. Find the expected value of the game to the bettor for each of these three bets.*

Solution:
There are 36 possible combinations of values on the dice. Therefore the probabilities are as follows: That

Dice Value	Probability
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

is, if H is the event of winning high, L is the event of winning low, and C the probability of winning with a seven, then $P(H) = 15/36 = P(L)$, and $P(C) = 6/36$.
We now note that generally,

$$u(x) = \text{net gain} \cdot P(W) - \text{net loss} \cdot P(L). \tag{2}$$

So we note that

$$u_{High}(x) = \begin{cases} 1 & \text{if } x \in \{8, 9, 10, 11, 12\} \\ -1 & \text{if } x \notin \{8, 9, 10, 11, 12\} \end{cases}$$

$$u_{Low}(x) = \begin{cases} 1 & \text{if } x \in \{1, 2, 3, 4, 5, 6\} \\ -1 & \text{if } x \notin \{1, 2, 3, 4, 5, 6\} \end{cases}$$

$$u_{Seven}(x) = \begin{cases} 4 & \text{if } x = 7 \\ -1 & \text{if } x \neq 7 \end{cases}$$

We will now expand the table to include the various $u(x)$ functions for given events.

Dice Value	Probability ($f(x)$)	$u_{\text{high}}(x)$	$u_{\text{Low}}(x)$	$u_7(x)$
2	1/36	-1	1	-1
3	2/36	-1	1	-1
4	3/36	-1	1	-1
5	4/36	-1	1	-1
6	5/36	-1	1	-1
7	6/36	-1	-1	4
8	5/36	1	-1	-1
9	4/36	1	-1	-1
10	3/36	1	-1	-1
11	2/36	1	-1	-1
12	1/36	1	-1	-1

Thus, $\sum_{x\in S} u(x)_{High} f(x) = -1/6 = \sum_{x\in S} u_{Low}(x) f(x) = \sum_{x\in S} u(x)_{Seven} f(x)$.

□

Question 3. *Let μ and σ^2 denote the mean and the variance of the random variable X. Determine*

$$E\left[\frac{(X-\mu)}{\sigma}\right] \tag{3}$$

and

$$E\left\{\left[\frac{(X-\mu)}{\sigma}\right]^2\right\} \tag{4}$$

Solution: For (3). Note that E is a linear operator, and since σ and μ are constants, (3) becomes $1/\sigma\,(E(X)-\mu)$, but since $E(X)=\mu$, this formula turns into 0.
For (4). note that it first turns into $1/\sigma^2\left(E\left(X^2-2X\mu+\mu^2\right)\right)$. Thus by making use of the that fact that E is a linear operator, we see that it turns into

$$\frac{1}{\sigma^2}\left(E(X^2)-E(2X\mu)+\mu^2\right) = \frac{1}{\sigma^2}\left(\sum_{x\in S} x^2 - \mu^2\right)$$

□

Question 4. *Let X equal the larger outcome when a pair of four sided dice is rolled. The pmf of X is*

$$f(x) = \frac{2x}{16}, \qquad x \in \{1,2,3,4\}.$$

Find the mean, the variance, and the standard deviation.

Solution:

$$\mu = \sum_{x=1}^{4}\left(x\cdot\frac{2x}{16}\right) = \frac{15}{4}$$

$$\sigma^2 = \sum_{x=1}^{4}\left(x^2\left(\frac{2x}{16}\right)\right) - \mu^2 = \frac{-25}{16}$$

$$\sigma = \sqrt{\sigma^2} = \frac{5}{4}$$

□