Combinatorics and Graph Theory

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Section 2.2

1. Prove that there are exactly 2^n different subsets of n elements without using the binomial theorem.

Proof. Let $A = \{a_1, a_2, ...a_n\}$, and let $S \subset A$. For a given $a_i \in A$, either $a_i \in S$ or $a_i \notin S$. That is, for a given a_i, there are two potential placements for it. Since there are n such a_i's, we have 2^n potential S's. □

2. Prove the absorbtion/extraction identity: If n is a positive integer and k is a nonzero integer, then

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Proof.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n\,(n-1)!}{k\,(k-1)!\,(n-k)!} = \frac{n}{k}\left(\frac{(n-1)!}{(k-1)!\,(n-k)!}\right) = \frac{n}{k}\binom{n-1}{k-1}$$

□

3. Prove the cancellation identity: If n and k are non-negative integers and m is an integer with $m \leq n$, then

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

Proof.

$$\begin{aligned}
&\binom{n}{k}\binom{k}{m} = \left(\frac{n!}{k!\,(n-k)!}\right)\left(\frac{k!}{m!\,(k-m)!}\right) \\
&= \left(\frac{n!}{m!\,(n-k)!\,(k-m)!}\right) \\
&= \left(\frac{n!}{m!\,(n-m)!}\right)\left(\frac{(n-m)!}{(k-m)!\,(n-k)!}\right) \\
&= \left(\frac{n!}{m!\,(n-m)!}\right)\left(\frac{(n-m)!}{(k-m)!\,(n-m-k+m)!}\right) \\
&= \binom{n}{m}\binom{n-m}{k-m}
\end{aligned}$$

□

5. Prove the hexagon identity:

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

Proof.

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$
$$\rightarrow \left(\frac{(n-1)!}{(k-1)!\,(n-k)!}\right)\left(\frac{n!}{(k+1)!\,(n-k-1)!}\right)\left(\frac{(n+1)!}{k!\,(n+1-k)!}\right) =$$
$$\left(\frac{(n-1)!}{k!\,(n-k-1)!}\right)\left(\frac{n!}{(k-1)!\,(n+1-k)!}\right)\left(\frac{(n+1)!}{(k+1)!\,(n-k)!}\right)$$
$$\rightarrow 1 = 1$$

□

8. In the Texas lottery, a gambler picks six numbers between 1 and 50; in the California lottery, a player chooses six numbers between 1 and 51. In both states, an entry wins something if at least 3 of the six numbers on the ticket are among the six numbers drawn. In which case is a player more likely to win something.

Solution:

The liklihood of correctly choosing three numbers from fifty is greater than the liklihood of choosing three from fifty-one.

□