Combinatorics and Graph Theory

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Section 2.1

1. In the C++ programming language, a variable name must start with a letter or the underscore character (_), and succeeding characters must be letters, digits, or the underscore character. Capital and lowercase letters are different. How many variable names having at most four letters can be formed in C++?

Solution

Consider the situation when the variable name has exactly four characters. For the first character, there are 53 possible choices. For the second through the fourth choices, there are 63 possible choices each. Thus there are $53\left(63^3\right)$ possible variables of four characters. For three characters, there are $53\left(63^2\right)$ possibilities, for two characters, there are 53(63) possibilities, and for one character, there is 53 possibilities. So the total possibilities for at most four characters is $53\left(1+63+63^2+63^3\right)$.

□

2. There are 29 teams in the National Basketball Association: 14 in the Western Conference, and 15 in the Eastern Conference.

a. Suppose each of the teams in the league has one pick in the first round of the NBA draft. How many ways are there to arrange the order of the teams selecting in the draft.

Solution

29!

□

b. Suppose that each of the first three positions in the draft must be awarded to one of the thirteen teams that did not advance to the playoffs that year. How many ways are there assign the first three positions in the draft?

Solution

Because order matters:

$$N =_{13} P_3 = \frac{13!}{10!} = 1716$$

□

c. How many ways are there for eight teams from each conference to advance to the playoffs, if order is unimportant?

Solution

$N =_{29} C_8 = 4292145$

□

d. Suppose that every team has three centers, four guards, and five forwards. How many ways are there to select an all-star team with the same composition from the Western Conference.

Solution In the Western Conference, there are 42 centers, 56 guards, and 70 forwards. Thus, there are ${}_{42}C_3$ ways of selcting centers, ${}_{56}C_4$ ways of selcting guards, and ${}_{70}C_5$ ways of selecting forwards. Therefore, since the selections are not completely independent, we use the multiplicitive principle, and see the number of ways of selecting a team is ${}_{42}C_3\left({}_{56}C_4\right)\left({}_{70}C_5\right) = 51,032,227,818,448,800$.

□

3. Compute the number of ways to deal each of the following five-card hands in poker. (*Note: to make the problem easier to solve, parts a-d will be solved in decending order.*)
d. Four of a kind. **Solution**
$N = 48(3) = 144$; because for a given rank there is one way of choosing the 4 of a kind and one in 48 ways of choosing the $5th$ card.

□

c. Straight Flush: both a straight and a flush. **Solution**
We know the only cards that can a first card for a straight-flush is $A - 10$, and because they must all be of the same suit; the number of possible straight flushes is $N = 40$.

□

b. Flush: all five cards have the same suit, but not in addition to being a straight. Make sure the solution counts for straights and flushes but do not include straight flushes.
Solution
${}_{13}C_5$ flushes per suit$\Rightarrow 4\left({}_{13}C_5\right)$ total flushes; so in order to know how many flushes we have without straight-flushes, we take $4\left({}_{13}C_5\right) - 40$.

□

a. Straight: the values of the cards form a sequence of consecutive integers. A jack has value 11, queen 12, king 13, and an ace 1 or 14. Furthermore, the cards in the straight can't all be of the same suit. Make sure the solution counts for straights and flushes but do not include straight flushes.
Solution We know that there are 40 choices for the first card. We then see that for every successive card in the straight, there are 4 possible outcomes. So we see that the number of straights is $40\left(4^4\right)$. Again, to get rid of the situations that we have straight flushes we take $40\left(4^4\right) - 40$.

□

4. A superstitious resident of Amarilla, Texas, always picks three even numbers and three odd numbers when playing the lottery. What fraction of all possible lottery tickets have this property?
Solution
Note that there are the number of evens equals the number of odds, and thus we want ${}_{25}C_3$ evens and ${}_{25}C_3$ odds. There are ${}_{50}C_6$ possible selections, so the fraction of 3 even - 3 odd lotto numbers to the total lotto numbers is $\left({}_{25}C_3\right)^2 / \left({}_{50}C_6\right) = 2300/6909$.

□

6. Let Δ be the difference operator: $\Delta\left(f(x)\right) = f\left(x+1\right) - f(x)$. Show that

$$\Delta\left(x^{\underline{n}}\right) = nx^{\underline{n-1}} \text{ (Part 1.)},$$

and use this to prove that

$$\sum_{k=0}^{m-1} k^{\underline{n}} = \frac{m^{\underline{n+1}}}{n+1} \text{ (Part 2).}$$

Proof. (Part 1.)

$$\begin{aligned}
&\Delta\left(x^{\underline{n}}\right) = \left(x+1\right)^{\underline{n}} - x^{\underline{n}} \\
&= \left(x+1\right)x\left(x-1\right)\left(x-2\right)\ldots\left(x-n+2\right) - x\left(x-1\right)\left(x-2\right)\ldots\left(x-n+1\right) \\
&= x\left(x-1\right)\left(x-2\right)\ldots\left(x-n+2\right)\left(\left(x+1\right)-\left(x-n+1\right)\right) \\
&= x\left(x-1\right)\left(x-2\right)\ldots\left(x-n+2\right)n \\
&= n\left(x\left(x-1\right)\left(x-2\right)\ldots\left(x-n+2\right)\right) \\
&= nx^{\underline{n-1}}
\end{aligned}$$

□

Proof. (Part 2.) Note that $x^{\underline{n}} = 1/(n+1)\Delta x^{\underline{n+1}}$. Then

$$\begin{aligned}
&\sum_{x=0}^{m-1} x^{\underline{n}} = \frac{1}{n+1}\sum_{x=0}^{m-1} \Delta x^{\underline{n+1}} \\
&= \frac{1}{n+1}\sum_{x=0}^{m-1}\left(\left(x+1\right)^{\underline{n+1}} - x^{\underline{n}}\right) \\
&= \frac{1}{n+1}\left(\left(1^{\underline{n+1}} - 0^{\underline{n+1}}\right) + \left(2^{\underline{n+1}} - 1^{\underline{n+1}}\right) + \ldots + \left(m^{\underline{n+1}} - \left(m-1\right)^{\underline{n+1}}\right)\right) \\
&= \frac{1}{n+1}\left(m^{\underline{n+1}} - 0^{\underline{n+1}}\right) \\
&= \frac{m^{\underline{n+1}}}{n+1}
\end{aligned}$$

□