Combinatorics and Graph Theory

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Section 1.6.1

2. Wite a nice proof of the fact that $R(1,k) = 1\ \forall k \in \mathbb{N}$.

Proof. We are looking for the smallest graph in which we are guaranteed to find a red K_1 or blue K_k. Regardless of how you color the (non-existant) edges of K_1, you can always find a red K_1. □

3. Write a nice proof of the fact that $R(2,k) = k$ s.t. $k \in \{k : k \in \mathbb{N} \text{ and } k \geq 2\}$.

Proof. To start, we show that $R(2,k) > k-1$. Consider the graph K_{k-1} colored entirely blue. Then obviously, there is no red K_2 or blue K_k, thus $R > k-1$. Now consider K_k. If any edge is red, then we have a red K_2, so the other condition would be if no edges were red. If no edges were red, then K_k would be completely blue, which is what we wanted. □

4. Prove that for positive integers p and q $R(p,q) = R(q,p)$.

Proof. The key feature of a Ramsey number is that it is the smallest number that guarantees that **any two coloring** of the edges of a complete graph guarantees a subgraph of a single color that is either a K_p or K_q; because of this definition, it follows naturally that the order in which we choose the colors is irrelevant. □

5. Prove the following theorem.

Theorem 1. *If $2 \leq p' \leq p$ and $2 \leq q' \leq q$, then $R(p',q') \leq R(p,q)$, where $R(p',q') = R(p,q) \Leftrightarrow p' = p$ and $q' = q$.*

Proof. By the definition of a Ramsey Number, $R(p,q)$ is the smallest natural number n such that K_n must contain either a red K_p or blue K_q. Therefore, if K_n contains a red K_p and $p' < p$, then K_n must also contain a red $K_{p'}$. The proof of the blue K_q and $K_{q'}$ follows identically to the previous.
For the second half of the theorem, we want to show that $R(p',q') = R(p,q) \Leftrightarrow p' = p$ and $q' = q$.
($\Leftarrow$)
Assume that $R(p',q') \neq R(p,q)$ but $p' = p$ and $q' = q$. Then w.o.l.g. $R(p',q') < R(p,q) \Rightarrow K_{R(p,q)}$ contains $K_{R(p',q')}$ as a subgraph $\Rightarrow p' < p$ and $q' < q$, which is a contradiction.
($\Rightarrow$)
Let $n = R(p,q)$. W.o.l.g. let $p' = p-1$ and $q' = q$. All colorings of K_n have a red K_p or blue K_q. Next, choose any vertex, and remove it along with its edges, yielding a K_{n-1}. Now I wish to show that all colorings of K_{n-1} contain either red K_{p-1} or blue K_q. To do this, we first take any such coloring of K_{n-1}, and we will assume no K_q. Add the missing vertex back in along with all of its red edges. The resulting graph must have a red K_p or a blue K_q; however, since we did not add any blue edges. there can still be no K_q, so we must have a red K_p. Therefore, either K_{n-1} had a red K_p or a red K_{p-1}. Eitherway, K_{n-1} had a K_{p-1} since K_{p-1} is a subgraph of K_p. Thus, $R(p-1,q) \leq n-1 < n = R(p,q)$. □

Section 1.6.2

1. Prove that $R(3,5) \geq 14$. The graph in Figure 1.103 will be helpful.

Proof. Consider the figure, which is the K_{13}. First we take a given vertex s, and look at the arrangement it has with the others as mod 13 arithmetic operating on the set $\{-6,-5,...,5,6\}$. If we are looking for a red K_3, then we note that s is connected to $s \pm 1$, $s \pm 5$, and since no two elements in $\{s, s \pm 1, s \pm 5\}$ differ by 1 or 5, we can't have a red K_3. If we are examining for a blue K_5, then we note we are looking at s's connections to $s \pm 2$, $s, \pm 3$, $s \pm 4$. So in order to show that no blue K_5 exists, we must try to construct one.

To do this, we need to try to figure out whether or not we can use both $s \pm 4$. The answer is no, because the distance, $d(-4,4) = 8 = -5$. So we can select at most either 4 or -4 to try to develop our K_5. Next, can we include both ± 6? Again, the answer is no, because $d(-6,6) = 12 = -1$. So we can have at most 6 or -6. Next we consider how many elements we can possibly take out of $\{\pm 2, \pm 3\}$. Here it is easily verified that we can take both ± 2 or ± 3, but not one from each.

So now consider what happens if we take $s \pm 2$. Then we cannot have $s \pm 3$ because $d(\pm 3, \pm 2) = 1$. We can't pick $s \pm 6$, because $d(\pm 6, \pm 2) = 8 = -5$. However, we can pick $s \pm 4$; however as mention previously, we can truly only take either 4 or -4, so we do not have enough vertices for a K_5. Similarly, if we take $s \pm 3$, then we can have $s \pm 2$ or $s \pm 4$ as $d(\pm 2, \pm 3) = d(\pm 4, \pm 3) = 1$. However we can take one of the $s \pm 6$; which results in the same problem of not having enough vertices to form a K_5.

$\therefore$ since K_{13} has been shown to have a situation in which we are not guaranteed either a red K_3 or blue K_5, $R(3,5) > 13 \Rightarrow R(3,5) \geq 14$. □

2. Use Theorem 1.45 and the previous exercise to prove that $R(3,5) = 14$.

Proof. Theorem 1.45 says that if $p \geq 2$ and $n \geq 2$ then

$$R(p,q) \leq R(p-1,q) + R(p,q-1).$$

By plugging in $p = 3$ and $q = 5$ in this theorem we see that $R(3,5) \leq R(3-1,5) + R(3,4)$; that is $R(3,5) \leq R(2,5) + R(3,4) \longrightarrow R(3,5) \leq 5 + 9 = 14$. Since $R(3,5) \leq 14$ and by the previous exercise $R(3,5) \geq 14$, then $R(3,5) = 14$. □

5. Use theorem 1.45 to prove theorem 1.46.

Theorem 2. *If $p = 3$, then for every integer $q \geq 3$*

$$R(3,q) \leq \frac{q^2+3}{2}$$

Proof. This proof will be completed by induction. For the base case, let $q = 3$. Then we have $R(3,3) \leq \left(3^2+3\right)/2 = 6$, which is true since $R(3,3) = 6$. Now we assume true $\forall q \leq n$, and now we must show this implies true for $n+1$.

$$R(3,n+1) \leq \frac{(n+1)+3}{2} = n^2+n+2$$

Now we use Theorem 1.45, and see that

$$R(3,n+1) \leq n+1+R(3,n).$$

Now we use the assumption that our theorem holds up to n, and see that $n+1+R(3,n) \leq n^2/2+n+5/2 < n^2+n+2$. □