# Combinatorics and Graph Theory

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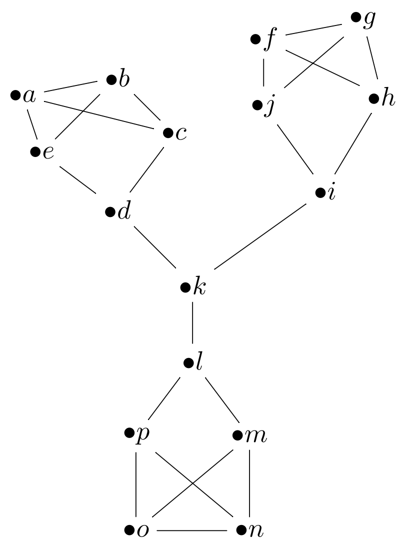
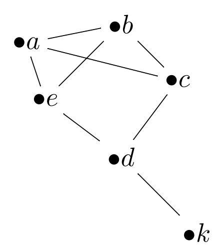
*From the text by Harris, Hirst, and Mossinghoff. Exercises from section 1.5 Section 1.5.1*

1. Determine whether the graph in figure 1.78 has a perfect matching. If not, explain why not.

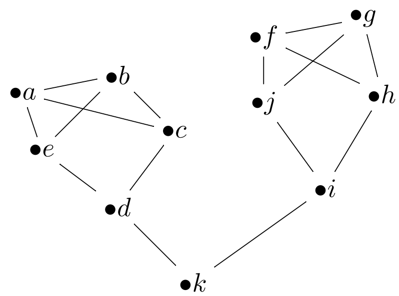



*Proof.* The above graph does not have a perfect matching. To illustrate this, consider first the subgraph $H =$




Now, we start to develop the matching $M$. We know to saturate $k$, that $(dk) \in M \Rightarrow (ed), (cd) \notin M$. Since $H$ is symmetric, we can choose w.l.g. we can let $(ac)(be) \in M$. So far, there is no a problem with the matching, so we can enlarge $H$ to encompass the next arm of $G$. So consider the newly enlarged $H$ below.

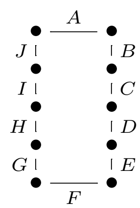
So now we let $H =$



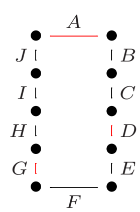

Recall that to this point $M = \{(dk), (ac), (be)\}$, and by considering the new $H$, we know immediately that $(ik) \notin M$, and it is easily seen that the size of the maximum (and maximal) matching for this branch is 2, thus creating a single unsaturated vertex. This is most easily seen by allowing $(ij) \in M \Rightarrow (jf) \notin M$, thus we can let $(fg) \in M$ which forces vertex $h$ to be unsaturated. Therefore, $G$ does not have a perfect matching. $\square$

2. Find the minimum size of a maximal matching in each of the following graphs.
a. $C_{10}$
b. $C_{11}$
c. $C_n$

*Solution*
*a.* Consider $C_{10}$ drawn as below




To start, let $A \in M \Rightarrow B, J \notin M$. The next step is to consider whether or not to let $C \in M$. Consider the consequence of including $D$ instead of $C$. If $D \in M$, the inclusion of $C$ would result in a non-matching situation, and since the inclusion of $D$ instead of $C$ is potentially smaller, it is safer to go with $D$. So at this point, $M$ contains two elements, namely $A$ and $D$. If we continue this patter we will arrive at the graph below (where the edges contained in the matching are colored red.)




The matching here is not maximal since the addition of edge $I$ results in a matching, therefore, in order to resolve this problem, $I \in M$. Therefore, $M(C_{10}) = \{A, D, G, I\}$, and $|M(C_{10})| = 4$.

*b.* Consider the graph of $C_{11}$. $|M(C_{11})| = 4$, by the same reasoning as above.
*c.* Consider the graph of $C_n$. By analyzing the situation above, and carrying out the procedure with graphs up to $C_{21}$, a pattern becomes apparant. If $n \in (3t-3, 3] \subset \mathbb{N}$, then $|M(C_n)| = t$.
*To prove the above, consider the restatement of the of problem below.*

**Theorem 1.** *The minimum size of a maximal matching for $C_n$ is $M(C_n) = t$ for $n \in (3t-3, 3t] \subset \mathbb{N}$.*

*Proof.* Suppose $\exists$ a maximal matching $\mathscr{T}$ with less than $t$ edges. Let $|\mathscr{T}| = r$. Note that $\exists$ at most two edges separating every edge in the matching. Let $P$ be a path in $G$ that starts and ends with edges in $\mathscr{T}$, and containing all edges in $\mathscr{T}$. Then length of this path is $|P| \leq 3r - 2 \leq 3(t-1) - 2$. Now consider $P' = G - P \Rightarrow |P'| + |P| = n$. This means that $|P'| = n - |P| \geq n - 3t + 5 \Rightarrow |P'| \geq n + 5 - 3t \Rightarrow |P'| \geq n + 5 - 3t > 5 + (-3) = 2 \Rightarrow \mathscr{T}$ can be made larger by the addition of atleast one edge and still be a matching.
$\therefore \exists$ no maximal matching $\mathscr{T}$ with less that $t$ edges.
$\therefore$ The minimum size of a maximal mathching for $C_n$ is $|M(C_n)| = t$. □

**Section 1.5.2**

2. For each of the following familes of sets, determine whether the condition of Theorem 1.33 is met. If so, then find an SDR. If not, then show how the condition is violated.

(Note: The theorem in question states that if we let $S_1, S_2, ..., S_k$ be a collection finite non empty sets, then the collection has an SDR iff $\forall t \in \{1, ..., k\}$, the union of any $t$ of these sets contains atleast $t$ elements.)
a.)$S_1 = \{1,2,3\}, S_2 = \{2,3,4\}, S_3 = \{3,4,5\}, S_4 = \{4,5\}, S_5 = \{1,2,5\}$
*Solution*

$$\left|\bigcup_i S_i\right| = 5 \Rightarrow \exists \text{ SDR.}$$

b.) $S_1 = \{1,2,4\}, S_2 = \{2,4\}, S_3 = \{2,3\}, S_4 = \{1,2,3\}$
*Solution*

$$\left|\bigcup_i S_i\right| = 4 \Rightarrow \exists \text{ SDR.}$$

c.) $S_1 = \{1,2\}, S_2 = \{2,3\}, S_3 = \{1,2,3\}, S_4 = \{2,3,4\}, S_5\{1,3\}, S_6 = \{3,4\}$
*Solution*

$$\left|\bigcup_i S_i\right| = 4 \Rightarrow \exists \text{ no SDR.}$$

Since there are two too few elements in the union.
d.) $S_1 = \{1,2,5\}, S_2 = \{1,5\}, S_3 = \{1,2\}, S_4 = \{2,5\}$
*Solution:*

$$\left|\bigcup_i S_i\right| = 3 \Rightarrow \text{ no SDR.}$$

e.) $S_1 = \{1,2,3\}, S_2 = \{1,2,4\}, S_3 = \{1,3,4\}, S_4 = \{1,2,3,4\}, S_5 = \{2,3,4\}$
*Solution:*

$$\left|\bigcup_i S_i\right| = 4 \Rightarrow \text{ no SDR.}$$

3. Let $G$ be a bipartite graph. Show that $G$ has a matching size atleast $|E(G)|/\Delta G$.

*Proof.* Let $X, Y$ be the partite sets in $G$. We wish to find a set $K$ such that $|K| = |M|$ and $K$ covers $G$. To do this, we use the construction found in the proof of the theorem 1.34 on page 67 of the text.
Define $W$ to be a subset of $X$ that contains only the vertices that are $M$-unsaturated. Note that $|M| = |X| - |W|$. Now let $A$ be the set of vertices of $G$ that can be reached via an $M$-alternating path from some vertex of $W$. Let $S = A \cap X$, and $T = A \cap Y$. Then the cover is $K = (X \backslash S) \cup T$.
Now $|K| = |X| - |S| + |T| = |X| - |W| = |M|$.
$\therefore |M| \Delta G \geq |E(G)|$ and we are done. □

5. Let $M_1$ and $M_2$ be matchings in a bipartite graph $G$ with partite sets $X, Y$. If $S \subset X$ is saturated by $M_1$ and $T \subset Y$ is saturated by $M_2$, show that there exists a matching, $M$, in $G$ that saturates $S \cup T$.

*Proof.* Let $H$ be a subgraph of $G$ such that $V(H) = V(G)$ and $E(H) = M_1 \cup M_2$. We will show that $M$ can be chosen in such a way as to saturate $S \cup T$. Since the degree of every vertex is at most two, then $H$ is a combination of paths, cycles, and singleton vertices.

If $M_1 \cup M_2$ creates a cycle $H$, then there is trivially a matching that is a subset of the cycle. Now we consider the possibilities of paths.

Suppose that $M$ does not saturate $S \cup T$, then without loss of generality, $\exists v \in S$ s.t. $v$ is unsaturated by $M$. Since $v \in S$, we know that $v$ is saturated by $M_1$, that is it has an edge $e$ coming out of it. Consider now the maximum length $M-$alternating path whose initial edge $e$. We know that the length of this path can not be odd, as it would create an $M-$augmenting path, which would be a contradiction.

So the path, if it exists, must be even. The edges in this path must alternate between $M_1$ and $M$. We also note that the terminal vertex $v_f \notin S$ because if $v_f \in S$ then we could extend by an edge of $M_1$ because $M_1$ saturates $S$. Since $v_f \notin S$, we could switch the edges included in $M$ to be the other edges on the path, and now $v \in S$ is saturated by $M$. □