Combinatorics and Graph Theory

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From the text by Harris, Hirst, and Mossinghoff. Exercises from section 1.4 Section 1.4.1

3. Senate committees C_1 through C_7 consist of members shown: $C_1 = \{$Adams, Bradford, Charles$\}$, $C_2 = \{$Charles, Davis, Eggers$\}$, $C_3 = \{$Davis, Ford$\}$, $C_4 = \{$Adams, Gardner$\}$, $C_5 = \{$Eggers, Bradford$\}$, $C_6 = \{$Howe, Charles, Ford$\}$, $C_7 = \{$Eggers, Howe$\}$. Use ideas of this section to determine the fewest number of meeting times that need by scheduled for these committees.
SOLUTION:
To model this situation as a graph, let us consider each committee as a vertex on a graph G. Then two vertices, C_i, C_j, are connected if and only if $C_i \cap C_j \neq \emptyset$, that is they contain none of the same members. Next, I will start to divide $V(G)$ into the disconnected subsets. For the first subset, I will consider C_1, and group it with all vertices $C_{i \neq 1}$ s.t. $C_{i \neq 1} \cap C_1 = \emptyset$, and will call this subset K_1, and we see that $K_1 = \{C_1, C_3, C_7\}$. Next, we will look at the next vertex not yet placed into a subset, this vertex happens to be K_2, and perform the same operation. $K_2 = \{C_2, C_4\}$. The next vertex not included is C_5, and I see that $K_3 = \{C_5, C_6\}$. Upon throwing a look at the graph, one sees this is the smallest number of such sets. So the solution is $\chi(G) = 3$.

4. In assigning frequencies to cellular phones, a zone gets a frequency to be used by all vehicles in the zone. Two zones that interfere must get different frequencies. How many different frequencies are required if there are six zones a, b, c, d, e, f, and zone a interferes with zone b; zone b with zones $a, c, d,$; zone c with b, d, e; d with b, c, e; e with c, d, f; and f with only e?
SOLUTION:
Suppose that each zone is a vertex on a graph G, and let two vertices share an edge only if they interfere. So realizing that while the interferences are listed in a way the implies a digraph, we see that there are no situations in which the interference is one directional, so we need not consider this graph as a digraph. Our edge set for this graph is $E(G) = \{(a,b),(b,c),(b,d),(c,d),(c,e),(d,e),(e,f)\}$. Next I must subdivide the vertex set V into subsets, such that for any $K_i, K_j \subsetneq V$ s.t. $i \neq j$, $K_i \cap K_j = \emptyset$ and $V = \bigcup_i (K_i)$. To do this, I will first place vertex a into a subset K_1, and since b is the only vertex adjacent to a, $b \in K_2$. Next, I realize that b is adjacent to two so far unaccounted for vertices, namely c, d. However, since c is adjacent to d only one of them can be in $K_1 \Rightarrow \exists K_3$. So w.l.o.g. let $d \in K_1$ and $c \in K_3$. Now note that e is adjacent to c, d which means it can't be in either K_1 or K_3, but e also is not connected to anything in K_2, so $e \in K_2$. Lastly, vertex f is connected only to e, so it can not belong to only K_2 but could belong to either K_1 or K_3, so w.l.o.g. we can say that $f \in K_1$. So we get the following subsets for our coloring

$$K_1 = \{a, d, f\}$$
$$K_2 = \{b, e\}$$
$$K_3 = \{c\}$$

By checking the graph, we can see that there is no way of removing one of the two colors, so $\chi(G) = 3$; which means that there must be at least three different frequencies to avoid interference.

5. When issuing seating assignments for third grade students, the teacher wants to be sure that if two students might interfere with one another, then they are assigned a different area of the room. There are six main trouble makers in the class: John, Jeff, Mike, Moe, Larry, and Curly. How many different areas are required in the room if John interferes with Moe and Curly; Jeff interferes with Larry and Curly; Mike interferes with Larry and Curly; Moe interferes with John, Larry, and Curly, Larry interferes with Jeff, Mike, Moe, and Curly; and Curly interferes with everyone?

SOLUTION:

Similarly to the last problem, the first thing I do is to realize that there are no one directed interferences, so I need not consider the graph of this situation as a digraph. Next then, I will let $a =$ John, $b =$ Jeff, $c =$ Mike, $d =$ Moe, $e =$ Larry, and $f =$ Curly, in order to simplify notation. Next I examine vertex a, and realize that a is connected to f and d, and also that d and f are also connected. This tells me immediately that there are at least three distinct and disjoint subsets (ie three colors). So let $a \in K_1$, $d \in K_2$, and $f \in K_3$. Note now that f is adjacent to all other vertices, so $K_3 = \{f\}$. So now, consider b, the next vertex not sorted so far, b is connected only to e and$f$, so I can then place $b \in K_1$, and since e is adjacent to b and not to d I can say that $e \in K_2$. Now I consider my remaining vertex c. c is connected only to f and e, and since $f \in K_3$ and $e \in K_2$, I know that $c \in K_1$. Therefore the minimum number of subsets needed is three, thus $\chi(G) = 3$, and we need three areas of the room for our trouble makers.

6. Prove that adding an edge to a graph increases its chromatic number by at most one.

Proof. $\chi(G)$ is dependent upon connections between vertices; such that the more connected the graph is, the greater the value of $\chi(G)$. Adding an edge to a given graph will connect two vertices, by the definition of an edge. Suppose now that the edge to be added is (v_1, v_2). So now there are two possibilities: either v_1, v_2 are the same color or they aren't.

If v_1 and v_2 are colored differently, then $\chi(G)$ will remain unchanged with the addition of the edge (v_1, v_2). If v_1 and v_2 are the same color, then the addition of (v_1, v_2) forces either v_1 or v_2 to change color. This change will result in a change in χ only if the vertex changing color is connected to vertex of every color contained in the graph. This potential change will only change $\chi(G)$ by 1, since otherwise a vertex would have to take on two colors. □

Section 1.4.2

1. Recall that avgdeg(G) denotes the average degree of vertices in G. Prove or give a counterexample to the following statement:

$$\chi(G) \leq 1 + \text{avgdeg}(G).$$

Proof. The statement is false. Consider graph with

$$V = \{a, b, c, d, e, f\}$$

and

$$E = \{(a, b), (a, f), (b, c), (b, d), (b, e), (c, d), (d, e), (e, f)\}.$$

In this case, $\chi(G) = 4$, and avgdeg$(G) = 2.5$, so since $4 > 3.5$, the proposed theorem can't be true. □

6. Prove that for any graph G of order n,

$$\frac{n}{\beta(G)} \leq \chi(G) \leq n + 1 - \beta(G)$$

Proof. I will first consider the right hand inequality. Let B be the largest independent set of G; thus $\beta = |B|$. If we consider the graph induced by B is an empty graph of β vertices with chromatic number equal to 1. Consider now the vertex set $A = V \backslash B$. Note now that $|A| = n - \beta$. Since $\exists$ no edge $e \in E$ s.t. $e = (v_1 \in B, v_2 \in A)$ and $A \cup B = V$, and A is connected $\Rightarrow$ that at most the graph induced by A is complete $\Rightarrow$ that the chromatic number of this induced graph is at most $|A| \Rightarrow \chi(G) \leq |A| \rightarrow \chi(G) \leq n + 1 - \beta$.

Next, I will consider next the left hand inequality. Suppose that we color the graph G with k colors. Then there are k independent sets; since G is finite, $\exists$ an independent set B s.t. $|B| \geq$ the size of every other independent set. This forces $k \leq n/|B|$ for any k. So if we select a specific B_0, then we have a specific k coloring where $k = \chi(G) \Rightarrow \chi(G) \leq n/\beta \Rightarrow n/\beta \geq \chi(G)$. □