Combinatorics and Graph Theory

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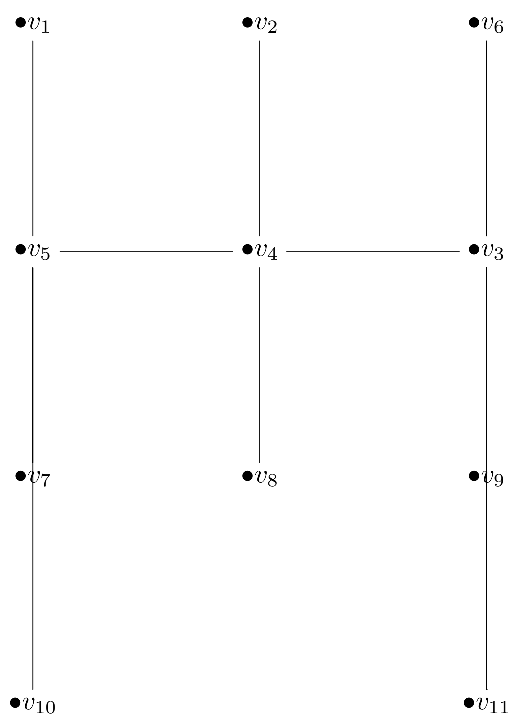
From the text by Harris, Hirst, and Mossinghoff. Exercises from section 1.2 (Cont. pt. 2)

Section 1.2.4

3. Draw and label the tree (T) whose Prfer sequence is $\sigma = 5, 4, 3, 5, 4, 3, 5, 4, 3$.



7. Use the matrix tree theorem to prove Caylee's Theorem.

Theorem 1. ***Caylee's Theorem***
There are n^{n-2} distinct labeled trees of order n.

Proof. In order to prove Caylee with the matrix tree theorem, we must choose to consider the graph K_n, since Caylee's theorem deals with the possible paths from vertex to vertex, where the matrix tree theorem works solely on the degree of the vertices. Therefore $\forall v_k \in V(G)$, $\deg(v_k) = n - 1$. Thus,

$$[A]_{i,j} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

$$[D]_{i,j} = \begin{cases} n-1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \ [D - A]_{i,j} = \begin{cases} n-1 & \text{if } i = j \\ -1 & \text{otherwise} \end{cases}$$

Thus, we can write $[D - A]_{i,j}$ as the $n \times n$ matrix

$$[D - A]_{i,j} = \begin{pmatrix} n-1 & -1 & -1 & -1 \\ -1 & n-1 & -1 & -1 \\ \vdots & \dots & \ddots & \vdots \\ -1 & -1 & -1 & n-1 \end{pmatrix}$$

The matrix tree theorem states that the number of unique spanning trees is equal to any cofactor of the above matrix. Thus we consider the cofactor of $i = j = 1$. So we see that

$$\det\left([D - A]\,(1|1)\right) = \begin{pmatrix} n-1 & -1 & -1 & -1 \\ -1 & n-1 & -1 & -1 \\ \vdots & \dots & \ddots & \vdots \\ -1 & -1 & -1 & n-1 \end{pmatrix}$$

where the matrix above is $n - 1 \times n - 1$. This matrix is capable of being made lower triangleable by two following operations. 1. $-R_{n-1} + R_i \forall i \in \{1, 2, \dots, n-2\}$. 2. $C_i + C_{n-1} \forall i \in \{1, 2, \dots, n-2\}$.
Operation 1 generates the matrix

$$\begin{pmatrix} n & 0 & 0 & -n \\ 0 & n & 0 & -n \\ \vdots & \dots & \ddots & \vdots \\ -1 & -1 & -1 & n-1 \end{pmatrix}$$

And step 2 generates

$$\begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ \vdots & \dots & \ddots & \vdots \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Thus we can compute the determinant of this matrix in a straight forward method by multiplying the diagonal entries. Since there are $n - 2$ n's, $\det\left(D - A(1|1)\right) = n^{n-2}$. ☐

Section 1.3.2

1. G is a planar graph of order 24, and it's regular of degree 3. How many regions are in the planar representation of G.

Solution:
$|V(G)| = 24$ and $\forall v_i \in V(G)$, $\deg(v_i) = 3$. $|E| = 1/2 \sum_i \deg(v_i) = 1/2(24)(3) = 36$, thus by Euler's formula, $r = 2 + |E| - |V| \Rightarrow 2 + 36 - 24 = 14$.

2. Let G be a connected planar graph of order less than 12. Prove that $\delta(G) \leq 4$.

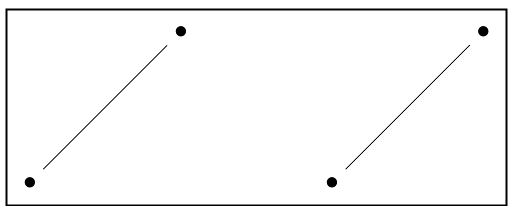
Proof. Let $n = |V(G)| < 12$, and let $|E(G)| = q$, and let $D = \sum_i \deg(v_i)$. Thus

$$D = 2q \leq 2\,(3n - 6)$$
$$\Rightarrow 2q \leq 6n - 12$$

Since $n < 12$. $D \leq 6n - 12 < 6n - n = 5n$. Thus, $\text{avedeg}(G) < 5 \Rightarrow \exists v_j \in V(G)$ s.t. $\deg(v_j) \leq 4$. □

3. Prove that Euler's formula fails for disconnected graphs.

Proof. Suppose that Euler's formula holds for disconnected graphs. Consider the following graph



Here obviously $|V| = n = 4$, $|E| = q = 2$, $r = 1$. Euler's formula says that $2 = n - q + r$; however, $2 \neq 4 - 2 + 1 = 3$. Thus, Euler's formula can't apply to disconnected graphs. □

4. Let G be a planar, triangle-free graph of order n. Prove that G has no more than $2n - 4$ edges.

Proof. Let $\mathscr{R}(G)$ be the set of regions of G. Since G is triangle-free, it contains no K_3 subgraphs. Thus $\forall R_i \in \mathscr{R}$, $\mathrm{b}(R_i) = m_i \geq 4 \Rightarrow$ there are at least four edges bordering any region $R_i \in \mathscr{R}$. Thus if $|\mathscr{R}| = r$, and because every edge is a border of precisely two regions, we have

$$2q = \sum_{i=1}^{r} m_i$$
$$\rightarrow 2q \geq 4r$$
$$\Rightarrow q \geq 2r$$

Using Euler's formula, $n - q + r = 2$, or re-written as I need it $r = 2 + q - n$, we see that

$$q \geq 2r$$
$$\rightarrow q \geq 2\,(2 + q - n) \quad \rightarrow q \geq 4 + 2q - 2n$$
$$\rightarrow -q \geq 4 - 2n$$
$$\rightarrow q \leq 2n - 4$$
$$\therefore G \text{ can have no more than } 2n - 4 \text{ edges.}$$

□

5. Prove that $\exists$ no bipartite graph with minimum degree at least four.

Proof. Suppose $\exists$ a bipartite graph, G, s.t. $\delta(G) \geq 4$. There $\exists$ $K_{3,3}$ subgraph of $G \Rightarrow G$ is not planar. This is a contradiction, therefore $\exists$ no bipartite graph G s.t. $\delta(G) \geq 4$. □

6. Let G be a planar graph with k components, prove that

$$n - q + r = 1 + k.$$

Proof. Let there be n_i vertices in component i, and define q similarly. Let j_i be the number of interior regions in the ith component. So $n = \sum_i n_i$, $q = \sum_i q_i$, and since for r we are looking at interior and the exterior regions, we see $r_i = j_i + 1$ and that $r = \sum_i j_i + 1$. So to examine the individual componets, we see that $n_i - q_i + r_i = 2 \rightarrow n_i - q_i + j_i + 1 = 2 \Rightarrow n_i + q_i + j_i = 1$. Now summing over the components we get

$$\sum_{i=1}^{k} (n_i - q_i + j_i) = \sum_{i=1}^{k} 1$$

$$n - q + \sum_{i=1}^{k} j_i = k$$

By adding one to both sides to generate the r on the left we get

$$n - q + r = 1 + k$$

□