Combinatorics and Graph Theory

Drew Robertson

March 22, 2007

From the text by Harris, Hirst, and Mossinghoff. Exercises from section 1.2 (cont.)

Section 1.2.3

1.

Theorem 1. *Every connected graph contains at least one spanning tree.*

Proof. By definition, a spanning tree of a graph G is a tree T such that $V(T) = V(G)$. Suppose G is a connected graph. Then G is either a tree or contains at least one cycle. If G is a tree, then G is a spanning tree of itself by exercise 2. Now suppose now that G is connected and is not a tree. W.L.O.G., suppose G contains a single cycle, C. Then $\exists e \in E(C \subsetneq G)$ s.t. $G - e$ results in a tree T. This tree was created by the removal of an edge from a cycle; therefore, T remains conncected and thus is a spanning tree. □

2.

Theorem 2. *A graph is a tree if and only if it is connected, and has exactly one spanning tree.*

Proof. G is a tree $\iff$ it is connected and contains no cycles $\iff \forall e \in E(G)$, $G - e$ is disconnected (since all edges in a tree are bridges) $\iff \exists!$ way in which the vertices can be connected $\iff G$ contains exactly one spanning tree. □

3.

Theorem 3. *Let G be a connected graph with n vertices and at least n edges. Let C be a cycle of G. If T is a spanning tree of G, then $\bar{T}$, the complement of T, contains atleast one edge of C.*

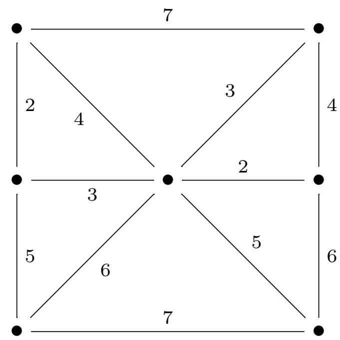
Proof. Let G be a connected graph with spanning tree T. Let the compliment of T be denoted as $\bar{T}$. W.L.O.G. assume that $\exists$ only a single cycle C. Suppose that $\forall$ edges e, $e \in C \Rightarrow e \notin \bar{T}$. Then $\forall e \in C$, $e \in T \Rightarrow$ that T is not a tree $\Rightarrow T$ can not be a spanning tree. This is a contradiction. Therefore, $\bar{T}$ must contain at least one edge of C. □

4. Let G be connected, and let $e \in E(G)$ Prove that e is a bridge if and only if it is in every spanning tree of G.

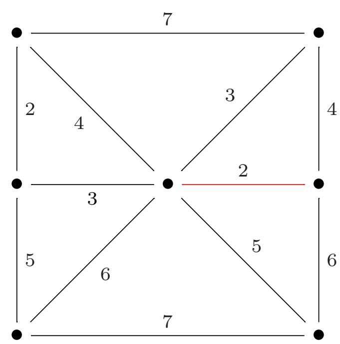
Proof. Let T be an arbitrary spanning tree. Since spanning trees are created by removing edges from the graph for which they span, T must remain connected, and the removal of any bridge would result in a disconnected graph. Therefore, a bridge $e \in E(G) \Rightarrow e \in E(T)$. □

5. Find the minimum weight spanning tree of the two figures below. Prove they are unique.

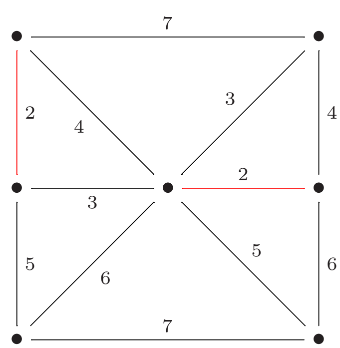
(A.)



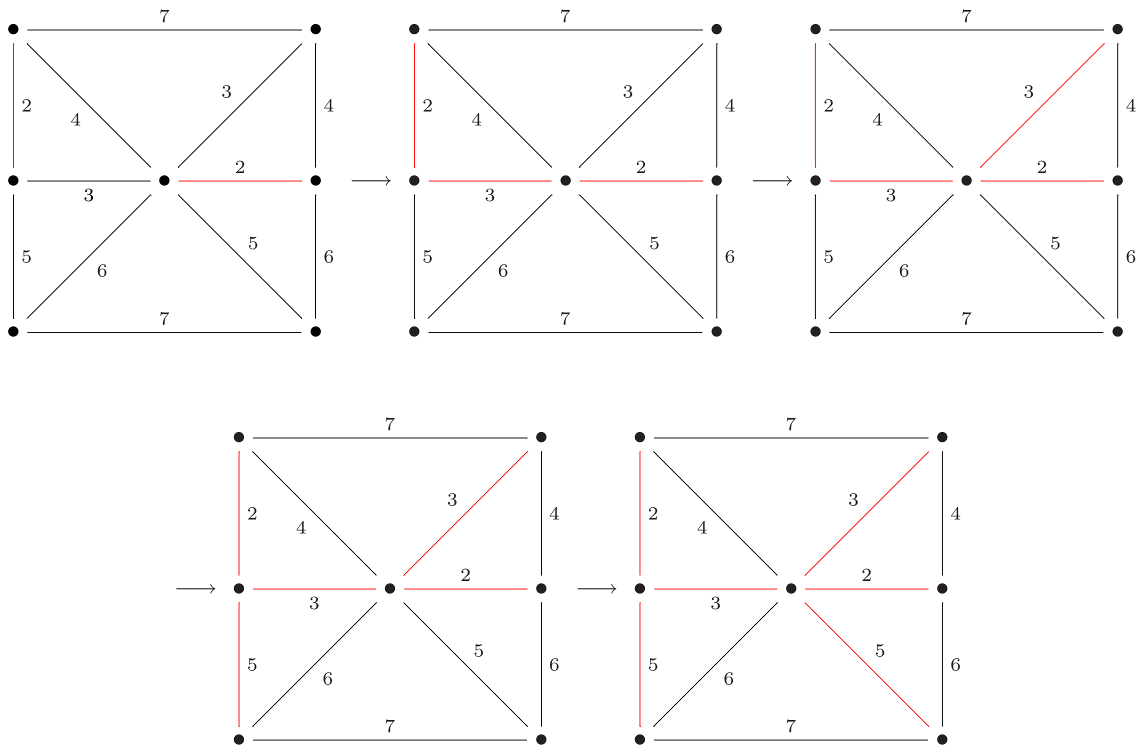
Proof. (A) By using Kruskal's algorithm, we will develop the minimum spanning tree. Step one is to select the minimum most weighted edge, and mark it. (I will shade all marked edges red in this process.) So after step one, I have



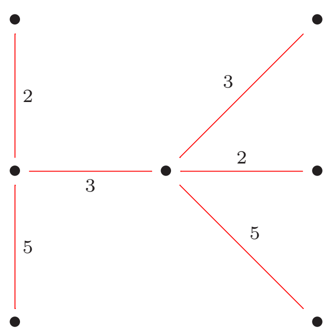
The next step is to select the next minimum most weighted edge, and mark it. So I will have



Next, we repeat this process, paying special attention as to not create a cycle. This process is done until all vertices have been included in the resulting subgraph. This process will result in the following markings

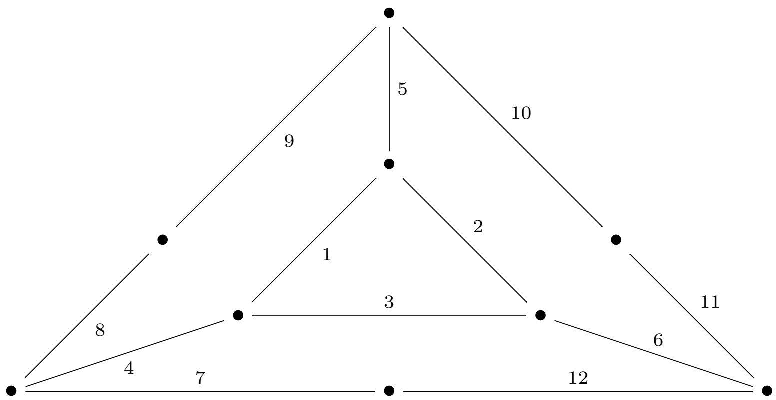


The red subgraph is the minimum weight spanning tree, and for clarification can be viewed without the cyclic edges. as follows

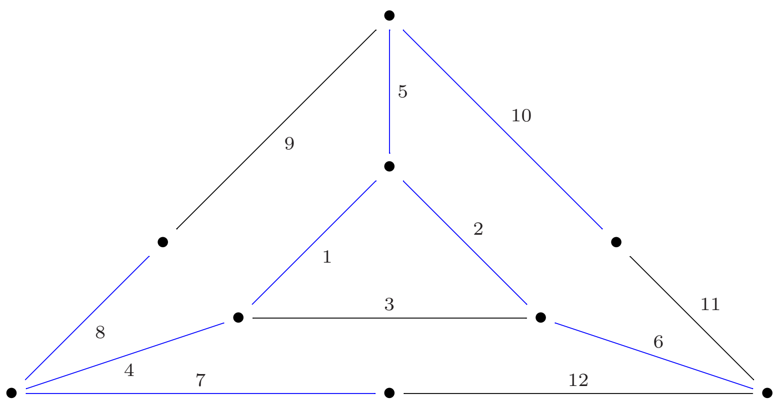


This spanning tree is unique, because with every iteration of the algorithm, we are forced to choose a non-cyclic edge of minimum weight. All edges that were removed from the graph to generate the spanning tree are not possible choices for inclusion in a minimum weight spanning tree by Kruskal's algorithm. □

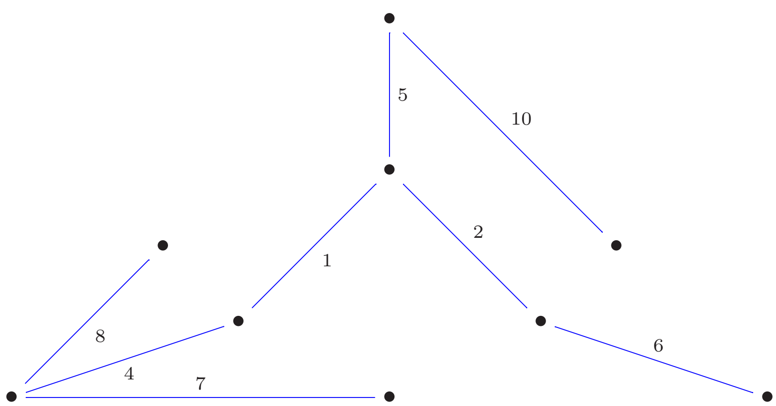
(B)



Proof. By using Kruskal's algorithm as before, we result with following subgraph (overlayed in blue)



or again for more clarity, the spanning tree can be viewed directly by removing the cyclic edges as



This spanning tree is unique because with Kruskal's algorithm forces us to choose the smallest non-cyclic edge on every itteration. Since there is no opportunity to choose one of the removed edges, it is imposible to use them to create a different minimum weight spanning tree. □