Combinatorics and Graph Theory

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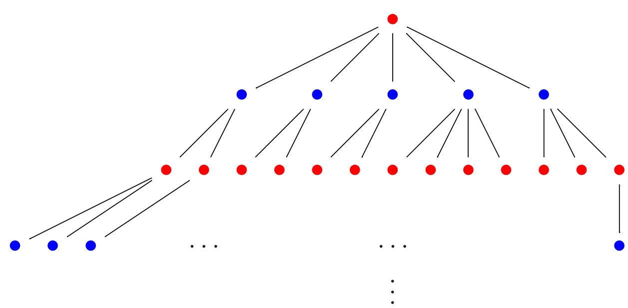
From the text by Harris, Hirst, and Mossinghoff. Exercises from section 1.2

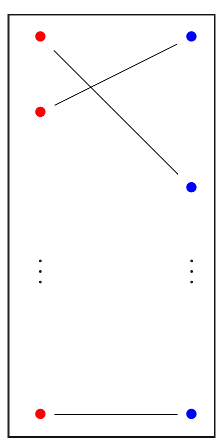
Section 1.2.1

3. Let T be a tree of order $n \geq 2$. Prove that T is bipartite.

Proof. Let T be defined as above. The tree can be viewed genearicaly as follows:



If we take every other row of vertices, and regroup them, we see we will have a column of red vertices and a column of blue vertices, with no like colored vertices connected, which implies we have a $X \subsetneq V$ and $Y \subsetneq V$ s.t. $X \cap Y = \emptyset$. This is the definition of bipartite, so we are done.



□

4. Graphs of the form $K_{1,n}$ are called stars. Prove that if $K_{r,s}$ is a tree, then it must be a star.

Proof. By previous exercise, we also know that $K_{r,s}$ is also bipartite $\Rightarrow \exists R, S$ s.t. $r \in R$ and $s \in S$. The notation in the book implies that by naming the graph K that it is a complete (in this case) bipartite graph. Now if we suppose that $r, s > 1$, the $\deg(r_i) = \deg(r_j) \forall r_i, r_j \in R$ and $\deg(s_m) = \deg(s_n) \forall s_m, s_n \in S$. Since K is complete and bipartite in order for K to be a tree either $|R|$ or $|S|$ must be equal to one. Therefore, r or s must be equal 1 which implies that $K_{r,s}$ is a star. □

Section 1.2.2

2. Suppose a tree T has an even number of edges. Show that atleast one vertex has even degree.

Proof. Suppose that $\forall v_i \in V$, $\deg(v_i)$ is odd. Since $|E|$ is even and

$$|E| = \frac{1}{2} \sum_i \left(\deg\left(v_i\right)\right)$$

then $1/2 \sum_i \deg(v_i)$ is even $\Rightarrow \sum_i \deg(v_i)$ is even. Since the sum is even and $\forall v_i \in V \; v_i$ is odd, there is an even number of vertices. Since T is a tree, the number of edges is $|E| = |V| - 1$, and since $|V|$ is even, $|E|$ is odd. CONTRADICTION! Therefore, if T is a tree having an even number of edges, $\exists \; v_k \in V$ for which $\deg(v_k)$ is even. □

3. Let T be a tree with max degree Δ. Prove that T has at least Δ leaves.

Proof. Let T be a tree with max degree Δ, then $\exists$ a vertex σ with Δ adjacent vertices. The vertices adjacent to σ are either a leaf, or they will spawn at least one more vertex a piece. And this process can be repeated indefinitely. If all the vertices are leaves it is clear that we would have Δ leaves. From the stand point of counting leaves, there is no difference between a vertex being a leaf and a vertex parenting a single vertex; because either way we have a single leaf. If these vertices spawn more than a single vertex, then we will obviously have more than Δ leaves. □

4. Let F be a forest of order n containing k connected components. Prove that F contains $n - k$ edges.

Proof. Let F be a forrest as defined by the theorem, we have $T_i \subsetneq F \forall i \in \{1, 2, \ldots, k\}$ trees in the forest. We also know that $\text{ord}(F) = \sum_i \text{ord}(T_i) = n$. By theorem 1.3 I know that $|E(T_i)| = \text{ord}(T_i) - 1$. Since the number of edges in a forest is the sum of the edges of the trees, $|E(F)|$ is given by

$$|E(F)| = \sum_{i=1}^{k} \left(\text{ord}(T_i) - i\right)$$
$$\Rightarrow |E(F)| = n - k$$

□

10. Let T be a tree of order $n > 1$. Show that the number of leaves is

$$2 + \sum_{\deg(v_i) \geq 3} (\deg(v_i) - 2)$$

where the sum is over all vertices of degree 3 or more.

Proof. If $n = 2$, then we have $2 + 0 = 2$ which is true by theorem 1.7. Suppose this is true for $n = k$. Then if we select a vertex $v_j \in V$ s.t. $\deg(v_j) > 2$ and $\exists$ no other vertices of degree greater than two below it. Then T has lost $\deg(v_j) - 2$ leaves. The number of leaves on the resulting tree still has form

$$2 + \sum_{\deg(v_i) \geq 3} (\deg(v_i) - 2)$$

where we are summing over all remaining vertices of degree greater than two, which is of the form required. Therefore, the original tree has $2 + \sum_{\deg(v_i) \geq 3} (\deg(v_i) - 2)$ leaves. $\square$