Combinatorics and Graph Theory

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From the text by Harris, Hirst, and Mossinghoff. Exercises from section 1.1

Section 1.1

4. Let G be a graph of order n and size strictly less than $n-1$. Prove that G is not connected.

Proof. Let $V = \{1, 2, \ldots n\}$. We know by assumption that $|E| < n-1$, so it suffices to show true for $|E| = n-2$, since if $|E| = n-2 \Rightarrow$ disconnected, so must $|E| = n-k$ for $2 < k \leq n$. Let $E = \{\epsilon_1, \epsilon_2, \ldots \epsilon_{n-2}$.

Suppose we start a path at $\delta_1 \in V$, δ_1 will be connected to some δ_2 by an edge $\epsilon_1 = \{\delta_1, \delta_2\}$. Then δ_2 will be connected to δ_3 by an edge $\{\delta_2, \delta_3\} = \epsilon_2$. By induction, we see that

$$\epsilon_{n-2} = \{\delta_{n-2}, \delta_{n-1}\}$$

This implies that $\exists$ no $\epsilon_k \in E$ s.t. δ_n is an end point on ϵ_k; which means that it is not possible for G to be connected. Therefore G is disconnected. □

6. Is it true that a graph having exactly two vertices of odd degree must contain a path from one to the other? Give proof or a counter example.

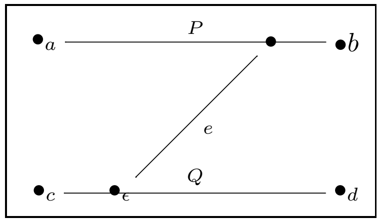
Proof. Let $\delta_1, \delta_2 \in V$ be of odd degree. Suppose $\exists$ no path from δ_1 to δ_2. Then $\exists$ subgraphs $F, H \subset G$ s.t. the $\sum_{v \in V(F)} \deg(v)$ and $\sum_{v \in V(H)} \deg(v)$ are odd $\Rightarrow 2|E|$ is odd. This is a contradiction! Therefore, $\exists$ a path between δ_1 and δ_2. □

7. Let G be a graph such that $\deg(v) \geq 2 \forall v \in V$. Prove that G contains a cycle.

Proof. Let $V = \{1, 2, \ldots, n\}$. W.O.L.O.G. we can let $\deg(v) = 2 \forall v \in V$, and suppose we start a path on this graph at vertex 1. At vertex 1 we can pick an edge to follow. By following this edge to a vertex, we can follow the other edge on it (so as to not retrace our path) to another vertex. We can continue this process, and since there are more edges than vertices we can conclude that by the n^{th} move we will have crossed a vertex more than once. That is the definition of a cycle. □

8. Let P and Q be two paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.

Proof. Suppose that P, Q do not share a common vertex. Then paths P and Q are disjoint. But since $P \subset G$ and $Q \subset G$, $\exists$ a path (in the simplest case, an edge) e s.t. e connects the paths of P and Q. This would imply that neither P or Q were of maximum length (since by traveling e the distance traveled would increase), therefore we have a contradiction. Therefore, P and Q must share atleast a common vertex.



□

9. Prove that an edge e is a bridge of G if and only if e does not lie on a cycle of G.

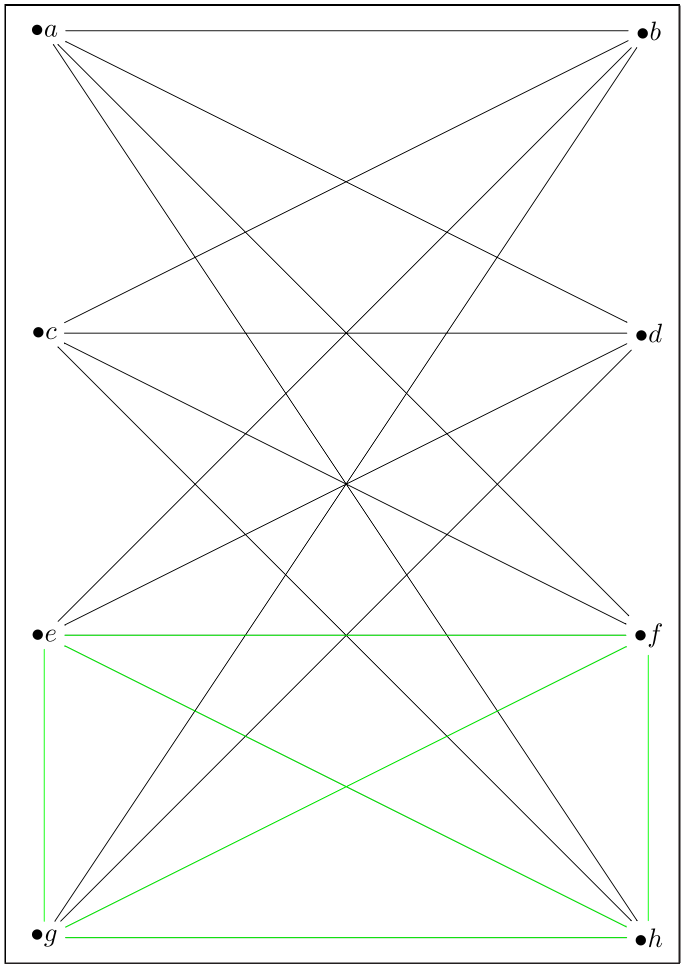
Proof. ($\Leftarrow$) Suppose e lies on a cycle. Then $G - e$ is connected $\Rightarrow$ e is not a bridge. Contradiction! $\therefore$ e can not lie on a cycle.
($\Rightarrow$) Since e is a bridge, then $G - e$ is disconnected. If e were on a cycle, $G - e$ would be connected. Therefore, e can not lie on a cycle. □

Section 1.1.3
1. If K_{r_1,r_2} is regular, prove that $r_1 = r_2$.

Proof. Let $G = K_{r_1,r_2}$. G is a bipartite graph. If G is regular, then every vertex in both subsets of V has to have the same number of incident edges. So W.O.L.O.G. suppose $r_2 > r_1$. Then the vertices of r_1 will have more incident edges than r_2, and therefore G would not be regular. Therefore, if G is to be regular as assumed, then $r_1 = r_2$. □

2. Determine whether K_4 is a subgraph of $K_{4,4}$. If yes, then exhibit it. If no, then explain why not.
Solution: K_4 is not a subset of $K_{4,4}$.

Proof. Consider the graph below. $K_{4,4}$ over layed with K_4.



Note that K_4 includes edges connecting the vertices on the same side of the figure where $K_{4,4}$ does not. Therefore, K_4 can not be a subgraph of $K_{4,4}$. □