Combinatorics and Graph Theory

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Theorem 1.
$$\sum_{\ell}\binom{\ell}{k}\binom{n+2}{2\ell+1}=\binom{n-k+1}{k}2^{n-2k+1}$$

Proof. The following lemma will be used in the proof.

Lemma 2.
$$(x+1)^n=\sum_{i=0}^{n}\binom{n}{i}(-1)^i$$

Since the sums together equal 2^n, then the equality implies each is equal to 2^{n-1}.

Proof. Let $x=-1$ Then $0^n=\sum_{i=0}^{n}\binom{n}{i}(-1)^i=\sum_{i\text{ even}}\binom{n}{i}-\sum_{i\text{ odd}}\binom{n}{i}\Rightarrow\sum_{i\text{ even}}\binom{n}{i}=\sum_{i\text{ odd}}\binom{n}{i}$. □

For $k=0$, the theorem reduces to
$$\sum_{\ell}\binom{n+2}{2\ell+1}=2^{n+1},$$
which is true by the lemma. We are going to induct upon n. Given an n, the inductive hypothesis is going to be:
$$\forall k\qquad\sum_{\ell}\binom{\ell}{k}\binom{n+2}{2\ell+1}=\binom{n-k+1}{k}2^{n-2k+1}.$$
Base case, $n=1$. If $n=1$, then $k=0$ or $k=1$. If $k=0$ then by the above, the theorem is true. If $k=1$, then $\ell=1$ since $k\le\ell\le(n+1)/2\Rightarrow$ LHS = RHS = 1. So the base case holds true. Assume the inductive hypothesis holds for $n\le t$. We shall show true for $t+1$. In other words we need to establish the following for all k.
$$\sum_{\ell}\binom{\ell}{k}\binom{(t+1)+2}{2\ell+1}=\binom{(t+1)-k+1}{k}2^{(t+1)-2k+1}$$
We first use the identity from Pascal's triangle to get
$$\text{LHS}=\sum_{\ell}\binom{\ell}{s}\binom{t+2}{2\ell}+\sum_{\ell}\binom{\ell}{s}\binom{t+2}{2\ell+1}$$
By the inductive hypothesis, the second sum can be evaluated
$$=\sum_{\ell}\frac{\ell}{k}\binom{\ell-1}{k-1}\binom{t+2}{2\ell}+\binom{t-k+1}{k}2^{t-2k+1}$$
$$=\sum_{\ell}\frac{\ell}{k}\binom{\ell-1}{k-1}\frac{t+2}{2\ell}\binom{t+1}{2\ell-1}+\binom{t-k+1}{k}2^{t-2k+1}$$
$$=\frac{t+2}{2k}\sum_{\ell}\binom{\ell-1}{k-1}\binom{t+1}{2\ell-1}+\binom{t-k+1}{k}2^{t-2k+1}$$

By letting $\ell' = \ell - 1$, we can use the inductive hypothesis on the left sum to yield

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\begin{aligned}
\text{LHS} &= \frac{t+2}{2k} \sum_{\ell'} \binom{\ell'}{k-1} \binom{t+1}{2\ell'+1} + \binom{t-k+1}{k} 2^{t-2k+1} \\
&= \frac{t+2}{2k} \binom{t-k+1}{k-1} 2^{t-2k+2} + \binom{t-k+1}{k} 2^{t-2k+1} \\
& 2^{t-2k+1} \left(\frac{t+2}{k} \binom{t-k+1}{k-1} + \binom{t-k+1}{k} \right) \\
& 2^{t-2k+1} \left(\frac{k}{k} \binom{t-k+1}{k-1} + \frac{t-k+2}{k} \binom{t-k+1}{k-1} + \binom{t-k+1}{k} \right) \\
& 2^{t-2k+1} \left(\binom{t-k+1}{k-1} + \binom{t-k+2}{k} + \binom{t-k+1}{k} \right) \\
& 2^{t-2k+1} \left(\binom{t-k+2}{k} + \binom{t-k+2}{k} \right) \\
& 2^{t-2k+2} \binom{t-k+2}{k} = RHS.
\end{aligned}
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□