M413
Introduction to Analysis I
Assignment XI

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Problem 1. *Show that* $\lim \sin\left(\frac{n\pi}{3}\right)$ *does not converge.*

Discussion 1. We will assume that $\lim \sin(n\pi/3)$ converges to L for some $L \in \mathbb{R}$, that is we will assume that $\lim(n\pi/3) = L$, and reach a contradiction. We will use the fact there are arbitrarily large values of $n \ni\colon \sin(n\pi/3) = 0$ and other arbitrarily large values of $n \ni\colon \lim \sin(n\pi/3) = \sqrt{3}/2$. So L will have to be arbitrarily close to $\sqrt{3}/2$ and arbitrarily close to 0 simultaneously. We will choose $\epsilon = \sqrt{3}/4$.

For our values of n for which $\sin(n\pi/3) = 0$, say n_1, note that $\sin(m\pi) = 0$ for $m \in \mathbb{N}$, which is equivalent to $\sin(3m\pi/3)$, so $n_1 = 3m$. For our arbitrary values of $n \ni\colon \sin(n\pi/3) = \sqrt{3}/2$, say n_2, note that $\sin(\pi/3 + 2\pi k) = \sqrt{3}/2$, for $k \in \mathbb{N} \Rightarrow \sin((1+6k)\pi/3) = \sqrt{3}/2$. So, we now see that $n_2 = 1 + 6k$.

Proof. Let $m, k \in \mathbb{N}$. Let $\epsilon = \sqrt{3}/4$. Assume that $\lim \sin(n\pi/3) = L$ for some $L \in \mathbb{R}$. Then, $\exists N \ni\colon n_1 = 3m, n_2 = 1 + 6k > N \Rightarrow |\sin(n_{1,2}\pi/3) - L| < \sqrt{3}/4$. Let $n > N$. So, we have

$$\begin{aligned}
&\sqrt{3}/2 = |\sqrt{3}/2| = \\
&|\sqrt{3}/2 - L + L| \leq |\sqrt{3}/2 - L| + |L| = |\sqrt{3}/2 - L| + |0 - L| = \\
&|\sin(n_2\pi/3) - L| + |\sin(n_1\pi/3) - L| < \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2};
\end{aligned}$$

that is $\sqrt{3}/2 < \sqrt{3}/2$ which is an obvious contradiction.
$\therefore \lim \sin(n\pi/3) \neq L$. Since L was arbitrarily chosen, $\lim \sin(n\pi/3)$ does not exist, and the sequence diverges. □

Problem 2. *Let* (s_n) *be a sequence of non-negative real numbers, and suppose* $\lim s_n = 0$. *Prove that* $\lim \sqrt{s_n} = 0$.

Discussion 2. $\forall \epsilon > 0$, we want a $N \ni\colon n > N \Rightarrow \left|\sqrt{s_n} - 0\right| < \epsilon$. $\left|\sqrt{s_n} - 0\right| = \left|\sqrt{s_n}\right| = \sqrt{s_n} < \epsilon \Rightarrow s_n < \epsilon^2$.

Proof. Let $\epsilon > 0$, then $\exists N \ni\colon n > N \Rightarrow |s_n - 0| < \epsilon^2 \Rightarrow s_n < \epsilon^2$. Let $n > N$. Then $\left|\sqrt{s_n} - 0\right| = \left|\sqrt{s_n}\right| = \sqrt{s_n} < \sqrt{\epsilon^2} = \epsilon$ □

Problem 3. *Show that if* $\lim u_n = u$, $\lim s_n = s, \lim t_n = t$, *then* $\lim (u_n + s_n + t_n) = u + s + t$.

Discussion 3. $\forall \epsilon > 0$ we want $N \ni\colon\ n > N \Rightarrow |(s_n + t_n + u_n) - (s + t + u)| < \epsilon$.

$$
\begin{aligned}
&|(s_n + t_n + u_n) - (s + t + u)| = |s_n + t_n + u_n - s - t - u| \\
&= |(s_n - s) + (t_n - t) + (u_n - u)| \leq |s_n - s| + |t_n - t| + |u_n - u| < \epsilon.
\end{aligned}
$$

This would suggest that we should use $\epsilon/3$ in place of just ϵ.

Proof. Let $\epsilon > 0$. Since $\lim s_n = s, \exists N_1 \ni\colon\ n > N_1 \Rightarrow |s_n - s| < \epsilon/3$. Since $\lim t_n = t, \exists N_2 \ni\colon\ n > N_2 \Rightarrow |t_n - t| < \epsilon/3$. Finally, since $\lim u_n = u, \exists N_3 \ni\colon\ n > N_3 \Rightarrow |u_n - u| < \epsilon/3$. Take $N = \max\{N_1, N_2, N_3\}$. Assume $n > N$. Then

$$|(s_n + t_n + u_n) - (s + t + u)| \leq |s_n - s| + |t_n - t| + |u_n - u| < 3\epsilon/3 = \epsilon$$

$\therefore$ If $\lim u_n = u$, $\lim s_n = s, \lim t_n = t$, then $\lim (u_n + s_n + t_n) = u + s + t$ $\square$

Problem 4. *Let* (t_n) *be a bounded sequence, ie,* $\exists M \ni\colon\ |t_n| \leq M \forall n$, *and let* (s_n) *be a sequence* $\ni\colon\ \lim s_n = 0$. *Prove that* $\lim (s_n t_n) = 0$.

Discussion 4. $\forall \epsilon > 0$ we want $N \ni\colon\ n > N \Rightarrow |s_n t_n - 0| < \epsilon$. Note that this implies directly that we want $|s_n t_n| = |s_n||t_n| < \epsilon$. So we will want to use ϵ/M as opposed to just an ϵ.

Proof. Let $\epsilon > 0$, then $\exists N \ni\colon\ n > N \Rightarrow |s_n| < \epsilon/M$. Let (t_n) be bounded $\ni\colon\ |t_n| \leq M$.
Then, $|s_n t_n - 0| = |s_n t_n| = |s_n||t_n| < M\epsilon/M = \epsilon$.
$\therefore$ If (t_n) is a bounded sequence, and (s_n) is a sequence $\ni\colon\ \lim s_n = 0$, then $\lim (s_n t_n) = 0$. $\square$