

M413
Introduction to Analysis I
Assignment X

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Problem 1. Prove $\lim(3n^2 + n)/(n^2 - 1) = 3$

Discussion 1. We want to show that $\forall \epsilon > 0, \exists N \ni n > N \Rightarrow |(3n^2 + n)/(n^2 - 1) - 3| < \epsilon$. We note that $(3n^2 + n)/(n^2 - 1)$ is only defined for $n > 1$ (since $n \in \mathbb{N}$.)

$$\left| \frac{3n^2 + n}{n^2 - 1} - 3 \right| = \left| \frac{3n^2 + n - 3n^2 + 3}{n^2 - 1} \right| = \left| \frac{n + 3}{n^2 - 1} \right| = \frac{n + 3}{n^2 - 1}.$$

We note now that we want $n + 3 < \text{something}$, and $n^2 - 1 > \text{something}$. Note that $n + 3 < n + 3n = 4n$. Since $n > 1$, $n^2 - 1 > n^2/2$ because $n^2 - 2 > 0$. So

$$\frac{n + 3}{n^2 - 1} < \frac{4n}{\frac{n^2}{2}} = \frac{8}{n} < \epsilon \Rightarrow n > \frac{8}{\epsilon}$$

So, we will let $N = \max\{1, 8/\epsilon\}$

Proof. Let $\epsilon > 0$, and $N = \max\{\sqrt{2}, 8/\epsilon\}$. Let $n > N$. Then

$$\left| \frac{3n^2 + n}{n^2 - 1} - 3 \right| = \left| \frac{n + 3}{n^2 - 1} \right| = \frac{n + 3}{n^2 - 1} < \frac{8}{n} < \epsilon.$$
$$\therefore \lim \frac{3n^2 + n}{n^2 - 1} = 3$$

□

Problem 2. Show that $\lim \sin\left(\frac{n\pi}{3}\right)$ does not converge.

Discussion 2. We will assume that $\lim \sin(n\pi/3)$ converges to L for some $L \in \mathbb{R}$, that is we will assume that $\lim(n\pi/3) = L$, and reach a contradiction. We will use the fact there are arbitrarily large values of n \ni $\sin(n\pi/3) = 0$ and other arbitrarily large values of n \ni $\lim \sin(n\pi/3) = \sqrt{3}/2$. So L will have to be arbitrarily close to $\sqrt{3}/2$ and arbitrarily close to 0 simultaneously. We will choose $\epsilon = \sqrt{3}/4$.

For our values of n for which $\sin(n\pi/3) = 0$, say n_1 , note that $\sin(m\pi) = 0$ for $m \in \mathbb{N}$, which is equivalent to $\sin(3m\pi/3)$, so $n_1 = 3m$. For our arbitrary values of n \ni $\sin(n\pi/3) = \sqrt{3}/2$, say n_2 , note that $\sin(\pi/3 + 2\pi k) = \sqrt{3}/2$, for $k \in \mathbb{N} \Rightarrow \sin((1 + 6k)\pi/3) = \sqrt{3}/2$. So, we now see that $n_2 = 1 + 6k$.

Proof. Let $m, k \in \mathbb{N}$. Let $\epsilon = \sqrt{3}/4$. Assume that $\lim \sin(n\pi/3) = L$ for some $L \in \mathbb{R}$. Then, $\exists N$ \ni $n_1 = 3m, n_2 = 1 + 6k > N \Rightarrow |\sin(n_{1,2}\pi/3) - L| < \sqrt{3}/4$. So, we have

$$\sqrt{3}/2 = |\sqrt{3}/2| = |\sqrt{3}/2 - L + L| \leq |\sqrt{3}/2 - L| + |L| = |\sqrt{3}/2 - L| + |0 - L| = |\sin(n_2\pi/3)| + |\sin(n_1\pi/3)| < \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

that is $\sqrt{3}/2 < \sqrt{3}/2$ which is an obvious contradiction.

$\therefore \lim \sin(n\pi/3) \neq L$. Since L was arbitrarily chosen, $\lim \sin(n\pi/3)$ does not exist, and the sequence diverges. \square