M413
Introduction to Analysis I
Assignment VII

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Question 1. *Let S be a non-empty bounded subset of $\mathbb{R}$. Then*
(a) Prove that $\inf(S) \leq \sup(S)$.
(b) What can we say about S if $\inf(S) = \sup(S)$*?*

Proof. (a)
Since S is bounded, the existance of $\inf(S)$ and $\sup(S)$ is assured. Since S is nonempty, $\exists s \in S$. By definition, $\forall s \in S, \inf(S) \leq s$ and $s \leq \sup(S)$, written another way $\forall s \in S$, $\inf(S) \leq s \leq \sup(S) \Rightarrow \inf(S) \leq \sup(S)$. □

Solution: (b)
$\inf(S) = \sup(S) \Leftrightarrow S = \{s\}$.

Proof. ($\Rightarrow$)
Let $\inf(S) = \sup(S)$. Suppose $S \neq \{s\}$. Since S is not a singleton, then minimally, $\exists t \in S \ni$: $S = \{s, t : s \neq t\}$. Without loss of generality, suppose $s > t$. Then by definition, $s = \max\{S\}$, and $t = \min\{S\}$. Thus by a previous theorem, $\max\{S\} = \sup(S)$ and $\min\{S\} = \inf(S) \Rightarrow s = t$. This is a contradiction, as we assumed $t \neq s$. Therefore $S = \{s\}$.
($\Leftarrow$)
Let $S = \{s\}$. Then by definition, $\max\{S\} = \min\{S\} = s$, and again by previous theorem, $\max\{S\} = \sup(S), \min\{S\} = \inf(S) \Rightarrow \sup(S) = \inf(S) = s$. □

□

Question 2. *Write the following sets in interval notation, and find the infimums and supremums of the sets:*

(a) $\{x \in \mathbb{R} : x < 0\}$
(b) $\{x \in \mathbb{R} : x^3 < 8\}$
(c) $\{x^2 : x \in \mathbb{R}\}$
(d) $\{x \in \mathbb{R} : x^2 < 8\}$

Solution:
(a) $S = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0) \Rightarrow \inf(S) = -\infty$, $\sup(S) = 0$.
(b) $S = \{x \in \mathbb{R} : x^3 < 8\} = \{x \in \mathbb{R} : x < 2\} = (-\infty, 2) \Rightarrow \inf(S) = 0$, $\sup(S) = 2$.
(c) $S = \{x^2 : x \in \mathbb{R}\} = [0, \infty) \Rightarrow \inf(S) = 0$, $\sup(S) = \infty$.
(d) $S = \{x \in \mathbb{R} : x^2 < 8\} = (-\sqrt{8}, \sqrt{8}) \Rightarrow \inf(S) = -\sqrt{8}$, $\sup(S) = \sqrt{8}$.

□