M413
Introduction to Analysis I
Assignment IV

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Theorem 1. *The following is a property of an ordered field.* $0 < 1$

Proof. Suppose that $1 < 0$. Then by property O4, $1 + 1 < 0 + 1 \Rightarrow 2 < 1$. This is a contradiction, as we know $1 < 2$. Therefore, $0 < 1$. □

Theorem 2. *The following property of an ordered field. If* $0 < a < b$*, then* $0 < b^{-1} < a^{-1}$.

Proof. Take $0 < a < b$, now using property vi of theorem 3.2, we see that $(a^{-1}b^{-1})0 < (a^{-1}b^{-1})a < (a^{-1}b^{-1})b \rightarrow 0 < (b^{-1}a^{-1})a < (a^{-1}b^{-1})b \rightarrow 0 < b^{-1}(a^{-1}a) \rightarrow 0 < b^{-1} \cdot -1 < a^{-1} \cdot 1 \rightarrow 0 < b^{-1} < a^{-1}$. □

Theorem 3. *Let* $a, b, c \in \mathbb{R}$*. Then* $|a + b + c| \leq |a| + |b| + |c|$.

Proof. Take $|a + b + c|$, and let $d = b + c$. Then $|a + b + c| = |a + d| \leq |a| + |d| = |a| + |b + c| \leq |a| + |b| + |c|$ Therefore

$$|a + b + c| \leq |a| + |b| + |c|.$$

□

Theorem 4. $\forall i \in \mathbb{N}$*, let* $a_i \in \mathbb{R}$*. Then* $a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$.

Proof. We wish to show that

$$\left| \sum_{i=1}^{n} a_i \right| \leq \sum_{i=1}^{n} |a_i| \, .$$

Base case of $n = 1$. Then clearly, $|a| = |a|$, so the statement holds for the base case. Suppose the statement holds $\forall n \leq m \in \mathbb{N}$. We wish to show true for $m + 1$. Note now that the sum on the first m terms is $|\sum_{i=1}^{m} a_i| \leq \sum_{i=1}^{m} |a_i|$. Now,

$$\left| \sum_{i=1}^{m+1} a_i \right| = \left| \sum_{i=1}^{m} a_i + a_{m+1} \right|. \qquad \text{Let } \beta = \sum_{i=1}^{m} a_i, \text{ then}$$

$$\left| \sum_{i=1}^{m} a_i + a_{m+1} \right| = \left| \sum_{i=1}^{m} \beta + a_{m+1} \right|. \qquad \text{Now since } \beta, a_{am+1} \in \mathbb{R}$$

$$|\beta + a_{m+1}| \leq |\beta| + |a_{m+1}| = \left| \sum_{i=1}^{m} a_i \right| + |a_{m+1}| \leq \sum_{i=1}^{m} |a_i + a_{m+1}| = \sum_{i=1}^{m+1} |a_i|$$

Therefore, the statement holds for all n. □