M413
Introduction to Analysis I
Assignment III

Drew Robertson

September 9, 2008

Problem 1. *Show that* $(2+\sqrt{2})^{1/2} \notin \mathbb{Q}$.

Proof. Let $a = (2+\sqrt{2})^{1/2}$. Then $a^2 = 2+\sqrt{2} \to a^2 - 2 = \sqrt{2} \to a^4 - 4a^2 + 2 = 0$. consider the equation

$$x^4 - 4x^2 + 2 = 0 \tag{1}$$

Suppose that (1) has a rational root. Then $\exists r \in \mathbb{Q}$ s.t. r is a solution to (1) and $r = p/q$ with $p, q \in \mathbb{Z}$ and $q \neq 0$. Then by the rational zero theorem, $p|1$ and $q|2$. Then, $p = \pm 1$ and $q = \pm 2$ or $q = \pm 1$. That is, $r = \pm 1/2$ or $r = \pm 1$. None of these work, so (1) has no rational solution, and since $(2+\sqrt{2})^{1/2}$ is a solution, $(2+\sqrt{2})^{1/2} \notin \mathbb{Q}$. □

Problem 2. *Show that* $(5-\sqrt{3})^{2/3} \notin \mathbb{Q}$.

Proof. Let $a = (5-\sqrt{3})^{2/3}$. Then $a^{3/2} = 5-\sqrt{3} \to a^{2/3} - 5 = -\sqrt{3} \to a^3 - 10a^{3/2} + 22 = 0$. Consider

$$x^3 - 10x^{3/2} + 22 = 0. \tag{2}$$

Suppose that $r \in \mathbb{Q}$ is a solution to (2). Then $\exists p, q \in \mathbb{Z}$ s.t. $q \neq 0$ and $p|1$ and $q|22$. Then $p - \pm 1$ and $q = \pm 1, \pm 2, \pm 11$, or ± 22. Since no r composed from any combination of these p/q's result in the (2) being true, then (2) has no rational solution. And since $(5-\sqrt{3})^{2/3}$ is a solution, $(5-\sqrt{3})^{2/3} \notin \mathbb{Q}$. □

Theorem 3. *The following statements hold* $\forall a, b, c \in \mathbb{R}$*.*

$$\begin{array}{ll} \textit{(i.)} & a + c = b + c \Rightarrow a = b \\ \textit{(ii.)} & (a) \cdot 0 = \\ \textit{(iii.)} & (-a)b = -(ab) \\ \textit{(iii b.)} & -(-a) = a \\ \textit{(iv.)} & -a(-b) = ab \\ \textit{(v.)} & ac = bc \textit{ and } c \neq 0 \Rightarrow a = b \\ \textit{(vi.)} & ab = 0 \Rightarrow a = 0 \textit{ or } b = 0 \end{array}$$

Problem 4. *Prove parts iii b, iv, and v of the above theorem.*

Solution: We will use the following lemma in the proofs of parts (iii b.) and (iv.).

Lemma 5. *Let* $a \in \mathbb{R}$*, then* $-1 \cdot a = -a \forall a \in \mathbb{R}$*.*

Proof. Note, that by definition of $-a$, $-a + a = 0$. It will suffice to show that $-1 \cdot a + a = 0$. Thus, note now that $a = 1 \cdot a$. Thus,

$$-1 \cdot a + a = -1a + (1 \cdot a) \rightarrow a(-1 + 1) = a(0) = 0 = -a + a$$

Therefore, $-a = -1 \cdot a$ □

Proof. (iii b.)
We need to show that $-(-a) = a$. Note first that by lemma, $-1 \cdot a = -a \forall a$. Thus, we apply the lemma twice, and see that $-(-a) = -1 \cdot (-1 \cdot a)$. Now by associativity of multiplication, we see that $-1 \cdot (-1 \cdot a) \rightarrow (-1 \cdot -1)a$, and since -1 is its own multiplicitive inverse, $-1 \cdot -1 = 1$, and thus $(-1 \cdot -1)a = a$. □

Proof. (iv.)
We need to show that $-a(-b) = ab$. First, we use the lemma twice again, to note that $-a(-b) = -1a(-1b)$, by making use of associativity we get $-1(a \cdot -1)b$, and by commutivity we note that this becomes $-1(-1a)b$, and by using the associativity again we get $-1 \cdot -1 \cdot ab = 1ab = ab$ by multiplicitive identity. □

Proof. (v.) We wish to show that if $c \neq 0$, then $ac = bc \Rightarrow a = b$. Suppose that $c \neq 0$. Then $\exists c^{-1} \ni: c \cdot c^{-1} = 1$. Thus we multiply both side by c^{-1}, and se $(ac)c^{-1} = (bc)c^{-1}$. And by associativity, we get $a(cc^{-1}) = b(cc^{-1})$ and by the definition of multiplicitive inverse, we see that $a \cdot 1 = b \cdot 1$, and my definition of multiplicitive identity, $a = b$. □

□