M413
Introduction to Analysis I
Assignment II

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September 6, 2008

Question 1. *Show that* $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31}$ *are irrational numbers.*

Proof. ($\sqrt{3}$)
Suppose that $\sqrt{3} \in \mathbb{Q}$, then $\exists m, n \in \mathbb{Z}$ s.t. $n \neq 0$ and $\sqrt{3} = m/n$. Then

$$3 = \frac{m^2}{n^2}$$
$$3n^2 = m^2,$$

then m^2 has either no factors, or an even number of factors of three, and n^2 has either no factors or an even number of factors of $3 \Rightarrow 3n^2$ has an odd number of three's. Thus, it is clear we have a contradiction. Therefore, $\sqrt{3}$ is irrational. □

Proof. ($\sqrt{5}$)
Suppose that $\sqrt{5} \in \mathbb{Q}$, then $\exists m, n \in \mathbb{Z}$ s.t. $n \neq 0$ and $\sqrt{5} = m/n$. Then

$$5 = \frac{m^2}{n^2}$$
$$5n^2 = m^2,$$

then m^2 has either no factors, or an even number of factors of five, and n^2 has either no factors or an even number factors of $5 \Rightarrow 5n^2$ has an odd number of five's. Thus, it is clear we have a contradiction. Therefore, $\sqrt{5}$ is irrational. □

Proof. ($\sqrt{7}$)
Suppose that $\sqrt{7} \in \mathbb{Q}$, then $\exists m, n \in \mathbb{Z}$ s.t. $n \neq 0$ and $\sqrt{7} = m/n$. Then

$$7 = \frac{m^2}{n^2}$$
$$7n^2 = m^2,$$

then m^2 has either no factors, or even factors of seven, and n^2 has either no factors or even number of factors of $7 \Rightarrow 7n^2$ has an odd number of seven's. Thus, it is clear we have a contradiction. Therefore, $\sqrt{7}$ is irrational. □

Proof. ($\sqrt{24}$)
Suppose that $\sqrt{24} \in \mathbb{Q}$, then $\exists m, n \in \mathbb{Z}$ s.t. $n \neq 0$ and $\sqrt{24} = m/n$. Then

$$24 = \frac{m^2}{n^2}$$
$$24n^2 = m^2,$$

then m^2 has either no factors, or even factors of twenty-four, and n^2 has either no factors or even number of factors of $24 \Rightarrow 24n^2$ has an odd number of seven's. Thus, it is clear we have a contradiction. Therefore, $\sqrt{24}$ is irrational. □

Proof. ($\sqrt{31}$)
Suppose that $\sqrt{31} \in \mathbb{Q}$, then $\exists m, n \in \mathbb{Z}$ s.t. $n \neq 0$ and $\sqrt{31} = m/n$. Then

$$31 = \frac{m^2}{n^2}$$
$$31n^2 = m^2,$$

then m^2 has either no factors, or even number of factors of 31, and n^2 has either no factors or even number of factors of $31 \Rightarrow 31n^2$ has an odd number of seven's. Thus, it is clear we have a contradiction. Therefore, $\sqrt{31}$ is irrational. □

Question 2. *Show that* $2^{1/3}, 5^{1/7}$, *and* $13^{1/4}$ *are not rational numbers.*

Proof. For the number $2^{1/3}$. Consider the equation $x^3 - 2 = 0$. Suppose that this equation has rational roots. Then $\exists$ a solution p/q s.t. $p, q \in \mathbb{Z}$ s.t. $q \neq 0$, p, q have no common factors, $p| - 2$, and $q|1$. Then $p = \pm 2$ or $p = \pm 1$ and $q = \pm 1$. It is clear that none of these is a solution to $x^3 - 2 = 0$, but $2^{1/3}$ is a solution. Therefore by the rational zero theorem, $2^{1/3} \notin \mathbb{Q}$. □

Proof. For the number $5^{1/7}$. Consider the equation $x^7 - 5 = 0$. Suppose this equation has rational zeros. Then $\exists$ a solution $p/q \in \mathbb{Q}$ s.t. p/q is in lowest terms, and $p| - 5$ and $q|1$. So $p = \pm 5$ or $p = \pm 1$ and $q = \pm 1$. However, none of these are solutions to the equation. But $5^{1/7}$ is a solution. Therefore, by the rational zero theorem, $5^{1/7} \notin \mathbb{Q}$. □

Proof. For the number $13^{1/4}$. Consider the equation $x^4 - 13 = 0$. Suppose this equation has rational zeros. Then $\exists$ a solution of the form $p/q \in \mathbb{Q}$ s.t. p/q is in lowest terms, and $p| - 13$ and $q|1$. So $p = \pm 13$ and $p = \pm 1$ or $q = \pm 1$. None of these are solutions to the equation. However, $13^{1/4}$ is a solution. Therefore, by the rational zero theorem, $13^{1/4} \notin \mathbb{Q}$. □