Abstract Algebra
Assignment 7

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Question 1. *Let $\mathbb{C}$ be the complex numbers, and $M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$. Prove that $\mathbb{C}$ and M are isomorphic under addition and that $\mathbb{C}^*$ and M^*, the non-zero elements of M, are isomorphic under multiplication.*

Proof. $(+)$
Let $\phi : M \to \mathbb{C}$ be given by $\phi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right) = a + bi$ Then, take $A, C \in M$ where $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, and $C = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ Then $\phi(A + C) = (a + bi + c + di) = (a + bi) + (c + di) = \phi(A) + \phi(B)$.
$\therefore \phi$ is an isomorphism, and hence $\mathbb{C}$ and M are isomorphic. $\square$

Proof. $(*)$
Let $A, C \in M^*$ be defined as above, and $\phi : M^* \to \mathbb{C}^*$ be defined as above. Then,

$$\begin{aligned}
&\phi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\right) \\
&= \phi\begin{pmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{pmatrix} \\
&= (ac - bd) + (ad + bc)i \\
&= ac + (bd + cb)i - bd \\
&= ac + bdi + cbi + bdi^2 \\
&= (a + bi)(c + di) \\
&= \phi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right)\phi\left(\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\right) \\
&= \phi(A)\phi(C)
\end{aligned}$$

$\therefore \phi$ is a homorphism, and hence M^* and C^* are isomorphic. $\square$