# Abstract Algebra
# Assignment 5

Drew Robertson

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*Chapter 4 revisted*

#3
List the subgroups of $\langle 20\rangle$ and $\langle 10\rangle$ in $\mathbb{Z}_{30}$.
*Solution:* The subgroups of $\langle 20\rangle$ are $\langle 10\rangle, \langle 0\rangle, \langle 20\rangle$, which also happen to be the subgroups of $\langle 10\rangle$.

□

#15

**Theorem 1.** *Let $G$ be an Abelian group and let $H = \{g \in G : |g| \mid 12\}$. Then $H$ is a subgroup of $G$.*

**Question 2.** *Is there anything special about 12 here? Would the proof be valid if 12 were replaced by some other integer $n$? State the general result.*

*Proof.* Let $G$ be a group, and $H \subset G$ where $H = \{g \in G : |g| \mid 12\}$. Take $a, b \in H$. Then clearly, $|a|, |b| \mid 12$, since $G$ is Abellian, $|ab| \mid 12 \Rightarrow ab \in H$. Since $|a| = |a^{-1}|$, and $|a|$ divides 12, $|a^{-1}|$ divides 12. Therefore, $H \leq G$.
*Solution:* The 12 is insignificant, and could have be any $n \in \mathbb{N}$. and the proof would have been identical. The general form is in the following theorem.

□

□

**Theorem 3.** *Let $G$ be an Abelian group and let $H = \{g \in G : |g| \mid n\}$. Then $H$ is a subgroup of $G$.*

*Proof.* Let $G$ be a group, and $H \subset G$ where $H = \{g \in G : |g| \mid n\}$. Take $a, b \in H$. Then clearly, $|a|, |b| \mid n$, and since $G$ is Abellian, $|ab| \mid n \Rightarrow ab \in H$. Since $|a| = |a^{-1}|$, and $|a|$ divides $n$, $|a^{-1}|$ divides $n$. Therefore, $H \leq G$.

□

#16
Find a collection of distinct subgroups $\langle a_1\rangle, \langle a_2\rangle, ... \langle a_n\rangle$ of $\mathbb{Z}_{240}$ with the property that $\langle a_1\rangle \subset \langle a_2\rangle \subset ... \subset \langle a_n\rangle$ with $n$ as large as possible. *Solution:* Note first that the biggest subgroup we can have is $\langle 1\rangle = \mathbb{Z}_{240}$. Then we have $\langle 2\rangle$ next. By taking multiples of the generator of the previous subgroups until we find one that has multiple equal to 240, we can build the following chain of subsets as follows

$$\langle 1\rangle \subset \langle 2\rangle \subset \langle 4\rangle \subset \langle 8\rangle \subset \langle 24\rangle \subset \langle 240\rangle.$$

□

#21
Let $G$ be a group and let $a \in G$.
(a) If $a^{12} = e$, what can be said about $|a|$?
(b) If $a^m = e$, what can be said about $|a|$?
(c) Suppose that $|G| = 24$ and that $G$ is cyclic. If $a^8 \neq e$ and $a^{12} \neq e$, show that $\langle a \rangle = G$.

*Solution:*
(a) $|a| = 12$
(b) $|a| = m$

$\square$

*Proof.* (c)
Let $G = \langle g \rangle$, then $a^k = g$. Let $a^8 \neq e$ and $a^{12} \neq e$. Since we know that $k \neq 8$ and $k \neq 12$ and $k|24$, $k = 1, 2, 3, 4$, or 6. However, since $a^k = e \Leftrightarrow |a| \mid k$, we know $|a| \neq 2, 3, 4$, or 6, because if it were either $a^8 = e$ or $a^{12} = e$. Therefore, there is only one option left, $k = 1$. So $a^1 = a = g \Rightarrow \langle a \rangle = \langle g \rangle$. $\square$