Abstract Algebra
Assignment 4

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Chapter 5, #'s 3, 24, 43, 50.

Question 1. *What is the order of each of the following permutations? (a)* (124)(357)
(b) (124)(3567)
(c) (124)(35)
(d) (124)(357869)
(e) (1235)(24567)
(f) (345)(245)

Solution:
(a) $|(124)(357)| = 3$
(b) $|(124)(3567)| = 12$
(c) $|(124)(35)| = 6$
(d) $|(124)(357869)| = 6$
(e) $|(1235)(24567)| = |(7356)(124)| = 12$
(f) $|(345)(245)| = |(52)(34)| = 2$

□

Question 2. *How many elements of order 5 are there in S_7.*

Solution: All elements of order 5 will be of five cycles. Thus, of the form $(a_1a_2a_3a_4a_5)$, then we notice that the number of ways of doing this are ${}_7P_5$, but we over counted, as some elements can be listed under different names, therefore, we note that we really have ${}_7P_5/5$.

□

Question 3. *Show that A_5 has 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2.*

Solution: For the number of elements of order 5, we want $\binom{5}{5}$ of possible elements, but the individual orderings matter, so we truly want $\binom{5}{5}5!$, but we must divide by 5 to avoid double counting, so we are left with $4! = 24$ elements.
For elements of order 3, we want $\binom{5}{3}$ of the possible elements, but order matters, so we have $\binom{5}{3}3!$, but we divide by 3 to avoid double counting; so we have $\binom{5}{3}3!/3 = 20$.
For order 15, we consider the ways of writing $\{a, b, c, d, e\}$ in the form of $(\gamma_1\gamma_2)(\gamma_3\gamma_4)$ where $(\gamma_1\gamma_2), (\gamma_3\gamma_4)$ are disjoint. Then by similar reasoning to the first two parts, we realize that for the first cycle, we have $\binom{5}{2}2!/2$ elements, and for the second cycle we have $\binom{3}{2}2!/2$ elements, and thus total number of elements is $\binom{5}{2}\binom{3}{2}$; however, we have over counted by 2, so we divide by two and see that we have $1/2\left(\binom{5}{2}\binom{3}{2}\right) = 15$.

□

Question 4. *A card shuffling machine always rearranges cards the same way relative to the order in which they were given to it. All the hearts arranged in order from ace to king were put into the machine, and then the shuffled cards were put into the machine to again be shuffled. If the cards emerged in the order* $10, 9, Q, 8, K, 3, 4, A, 5, J, 6, 2, 7$*, in what order were the cards after the first shuffle?*

Solution:

Order of Cards after Shuffle n													
$n = 0$:	A	2	3	4	5	6	7	8	9	10	J	Q	K
$n = 2$:	10	9	Q	8	K	3	4	A	5	J	6	2	7

Suppose that A was in the second position after the first shuffle, that is that $A \mapsto 2$. Then we know that the card in first position is moved to second. Thefore, we conclude that the $9 \mapsto 1 \Rightarrow 10 \mapsto 9 \Rightarrow 5 \mapsto 10 \Rightarrow J \mapsto 5 \Rightarrow K \mapsto J \Rightarrow 6 \mapsto K \Rightarrow 7 \mapsto 6 \Rightarrow 3 \mapsto 7 \Rightarrow 4 \mapsto 3 \Rightarrow Q \mapsto 4 \Rightarrow 8 \mapsto Q \Rightarrow 2 \mapsto 8$. So we re-write the table with the missing row as follows:

Order of Cards after Shuffle n													
$n = 0$:	A	2	3	4	5	6	7	8	9	10	J	Q	K
$n = 1$:	9	A	4	Q	J	7	3	2	10	5	K	8	6
$n = 2$:	10	9	Q	8	K	3	4	A	5	J	6	2	7

□