Abstract Algebra
Assignment 2

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Chapter 3, numbers 1,2,4,14,15,21,52, and 54.

Question 1. *For each group in the following list, find the order of the group and the order of each element in the group. What relation do you see between the order of each element in the group. What relation do you see between the orders of the elements of the group and the order of the group?*

Solution: $|Z_{12}| = 12$ $|0| = 0$, and $|n| = 12 - n \forall n > 0$.

$U(10) = \{1, 3, 7, 9\} \Rightarrow |U(10)| = 4$. $|1| = 1, |3| = 4, |7| = 4, |9| = 2$

$U(12) = \{11, 7, 5, 1\} \Rightarrow |U(12)| = 4$. $|1| = 1, |5| = 2, |7| = 2, |11| = 2$.

$U(20) = \{19, 17, 13, 11, 9, 7, 3, 1\}$. $|U(20)| = 8$. $|1| = 1, |3| = 4, |7| = 4, |9| = 2, |11| = 2, |13| = 4, |17| = 4, |19| = 2$.

$|D_4| = 8$. $|R_0| = 1, |R_{90}| = 4, |R_{180}| = 2, |R_{270}| = 4, |H| = 2, |V| = 2, |D| = 2, |D'| = 2$

We note that if G is a group and $a \in G$, then $|a| \mid |b|$.

□

Question 2. *Let Q be the group of rational numbers under addition and let Q^* be the group of non-zero rational numbers be the group of non-zero rational numbers under multiplication. In Q list the elements of in $\langle 1/2 \rangle$. In Q^*, list the elements of $\langle 1/2 \rangle$.*

Solution: For Q, $\langle 1/2 \rangle = \{n(1/2) : n \in \mathbb{Z}\}$.
For Q^*, $\langle 1/2 \rangle = \{(1/2)^n : n \in \mathbb{Z}\}$.

□

Theorem 3. *In any group, an element and its inverse have the same order.*

Proof. Let G be a group, and let $a, a^{-1} \in G$. Let $|a| = n$ and let $|a^{-1}| = m$. Then n is the least integer s.t. $a^n = e$, similarly for $m, \left(a^{-1}\right)^m = e$. So $(a^{-1})^n = (a^n)^{-1} = e^{-1} = e$. So $m \leq n$. So $a^m = a^{(-(-m))} = a^{-1 \cdot -m} = (a^{-1})^{(m)(-1)} = e^{-1} = e$. So $n \leq m$. So $n = m$, and $|a| = |a^{-1}|$. □

Theorem 4. *If H, K are subgroups of G, then $H \cap K$ is a subgroup G.*

Proof. First note that since H, K are subgroups, $H \cap K$ inherits the operation of G, Let $a, b \in H \cap K$. Then $a \in H, a \in K$ and $b \in H, b \in K$. Then because $H \leq G$ and $K \leq G, ab^{-1} \in H$ and $ab^{-1} \in K$. Thus $ab^{-1} \in H \cap K$. Therefore, $H \cap K$ is a subgroup of G. ☐

Theorem 5. *Let G be a group. Show that $Z(G) = \bigcap_{a \in G} C(a)$,*

Proof. Let G be a group, and let $Z(G)$ be the center of the group, ie $Z(G) = \{a \in G : ax = xa \forall x \in G\}$. $\beta \in Z(G) \Leftrightarrow \forall x \in G \Leftrightarrow \beta x = x\beta \Leftrightarrow \beta \in C(x) \forall x \in G \Leftrightarrow \beta \in \bigcap_{x \in G} C(x)$. ☐

Question 6. *Must the centralizer of an element be Abelian?*

Solution: No. Consider $C(a) = \{g \in G : ag = ga\}$. If $g, h \in C(a)$ and $g \neq h$ and neither g or h equals a. Then g need not commute with h.

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Theorem 7. *Let G be a finite group with more than one element. Show that G has an element of prime order.*

Proof. Let G be a group $|G| > 1$, and let $x \in G$ s.t. $x \neq e$. Let $|x| = k$. Then either k is prime or it is divisible by a prime. If k is prime, we're done. Suppose k is not prime. Then $\exists m, p$ s.t. $m \in \mathbb{Z}$, p is prime, and $mp = k$. Then we note that $x^k = e = x^{mp} = (x^m)^p \Rightarrow |x^m| = p$. Therefore, for a finite group G where $|G| > 1, \exists$ an element in G with prime order. ☐

Theorem 8. *Let G be a finite Abelian group and let $a, b \in G$. Prove the set $\langle a, b \rangle = \{a^i b^j : i, j \in \mathbb{Z}\}$ is a subgroup of G. What can we say about $|\langle a, b \rangle|$ in terms of $|a|, |b|$.*

Proof. Since $a, b \in \langle a, b \rangle, \langle a, b \rangle \neq \emptyset$. Let $a^m b^z, a^n b^y \in \langle a, b \rangle$. Then $a^m b^z (a^n b^y)^{-1} = a^m b^z (a^{-n} b^{-y})$. Now, because G is Abelian, $a^m b^z (a^{-n} b^{-y}) = a^{m-n} b^{z-y} \in \langle a, b \rangle$. So by previous theorem, $\langle a, b \rangle$ is a subgroup. ☐