Abstract Algebra
Assignment 1

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August 26, 2008

Theorem 1. *Let G be a group with the following property: When ever $a, b, c \in G$ and $ab = ca$, then $b = c$. Then G is Abelian.*

Proof. Suppose G is a non-Abelian, and let $a, b, c \in G \ni$: $ab = ca \Rightarrow b = c$ and $a \neq e$. If we multiply both sides of the equation by a^{-1}, we get $aa^{-1}b = a^{-1}ca \rightarrow b = a^{-1}ca$. This is an obvious contradiction, so G must be Abelian. $\square$

Theorem 2. *Let G be a group. Then G is Abelian if and only if $\forall a, b \in G$, $(ab)^{-1} = a^{-1}b^{-1}$*

Proof. $(\subset)$
Suppose G is Abelian. Then $(ab)^{-1}(ab) = 1 = (a^{-1}b^{-1})(ab)$. Therefore, $(ab)^{-1} = a^{-1}b^{-1}$. $\square$

Proof. $(\supset)$
Suppose $(ab)^{-1} = a^{-1}b^{-1}$, and suppose G is non-Abelian. Then $aa^{-1}bb^{-1} = a(ab)^{-1}b \rightarrow e = a(ab)^{-1}b$, which is a contradiction, because clearly if $a \neq e$ or $b \neq e$ then $e \neq a(ab)^{-1}b$. Therefore G is Abelian. $\square$

Theorem 3. *Let G be a group. Then $\forall a \in G$, $(a^{-1})^{-1} = a$.*

Proof. Let $a \in G$. Then because G is a group, $\exists b \in G$ s.t. $ba = ab = e$. Let $b = a^{-1}$. Now, since $b \in G, \exists c \in G$ s.t. $cb = bc = e$; note then that $c = a$. Therefore, $(a^{-1})^{-1} = a$. $\square$

Theorem 4. *For any elements a, b of a group G and any integer n, $(a^{-1}ba)^n = a^{-1}b^n a$.*

Proof. This proof will be done in three cases.
Case 1. Suppose $n = 0$, then we have $e = e$, thus the statement is true.
Case 2. Suppose $n > 0$. We will use induction to prove this.
Let $n = 1$. Then we trivially have $a^{-1}ba = a^{-1}ba$, thus the statement holds for the base case. Suppose the statement holds $\forall n \leq m - 1$, we wish to show true for $n = m$. That is, we wish to show that

$$\left((a^{-1}ba\right)^m = a^{-1}b^m a.$$

Note that $LHS = (a^{-1}ba) \cdot (a^{-1}ba)^{m-1}$. Thus by two applications of associativity, we see that $LHS = a^{-1}b^m a = RHS$.
Case 3. Suppose $n < 0$. Then note that $e = (a^{-1}ba)^n(a^{-1}ba)^{-n} \rightarrow e = (a^{-1}ba)^n(a^{-1}b^{-n}a)$. The multiplicitive inverse of the second factor on the RHS is $a^{-1}b^n a$, which thus gives us the equality $a^{-1}b^n a = (a^{-1}ba)^n$. $\square$

Question 5. *Give an example of a group with 105 elements. Give two examples of groups with 44 elements.*

Solution: $\mathbb{Z}_{105}$ has 105 elements. $\mathbb{Z}_{44}$ has 44 elements.

$\square$

Theorem 6. *Let $a, b \in G$. If $(ab)^2 = a^2b^2 \in G$, then G is Abelian.*

Proof. Suppose G is non-Abelian, and $(ab)^2 = a^2b^2$. Note then that $(ab)^2 = (ab)(ab) = abab$. Thus we have a contradiction, since because G is non-Abelian, $(ab)^2 \neq abab$. Therefore, G is Abelian. □

Theorem 7. *Let G be a group with the property that the square of every element is the identity. Then the group G is Abelian.*

Proof. Let G be a group s.t. $\forall a \in G, a^2 = e$. Let $a, b \in G$. Then $ab \in G$ by closure. Thus $(ab)^2 = e = a^2b^2 = b^2a^2$. Therefore, G is Abelian. □