Algebra of Functions

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This presentation was created with Microsoft PowerPoint 2002 (Office XP) and Design Science MathType 5.

Objectives

- To define the sum, difference, product, and quotient of functions.
- To form and evaluate composite functions.

Basic function operations

- ***Sum*** $(f+g)(x)=f(x)+g(x)$
- ***Difference*** $(f-g)(x)=f(x)-g(x)$
- ***Product*** $(f\bullet g)(x)=f(x)\bullet g(x)$
- ***Quotient*** $(f/g)(x)=\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)},\ g(x)\neq 0$

Function, domain, & range

- The **domain** of a function is the set of all “first coordinates” of the ordered pairs of a relation.
- The **range** of a function is the set of all “second coordinates” of the ordered pairs of a relation.
- A relation is a function if all values of the domain are unique (they do not repeat).
- A test to see if a relation is a function is the **vertical line test**.
  - If it is possible to draw a vertical line and cross the graph of a relation in more than one point, the relation is not a function.

Example 1

- Find each function and state its domain:

$$f(x)=\sqrt{x+1};\ g(x)=\sqrt{x-1}$$

  - $f+g$ $\quad (f+g)(x)=\sqrt{x+1}+\sqrt{x-1};\ D_{f+g}=\{x : x\geq 1\}$
  - $f-g$ $\quad (f-g)(x)=\sqrt{x+1}-\sqrt{x-1};\ D_{f-g}=\{x : x\geq 1\}$
  - $f\cdot g$ $\quad (f\cdot g)(x)=\left(\sqrt{x+1}\right)\left(\sqrt{x-1}\right)=\sqrt{x^2-1};\ D_{f\cdot g}=\{x : x\geq 1\}$
  - f/g $\quad (f/g)(x)=\dfrac{\sqrt{x+1}}{\sqrt{x-1}};\ D_{f/g}=\{x : x>1\}$

Example 2

- The efficiency of an engine with a given heat output, in calories, can be calculated by finding the ratio of two functions of heat input, *D* and *N*, where **$D(i) = i - 5700$** and **$N(i) = i$.**
  - Write a function for the efficiency of the engine in terms of heat input (i), in calories.

$$E(i) = \frac{i - 5700}{i}$$

➢ Find the efficiency when the heat input is 17,200 calories.

$$E(17{,}200) = \frac{17{,}200 - 5700}{17{,}200} \approx 0.67$$

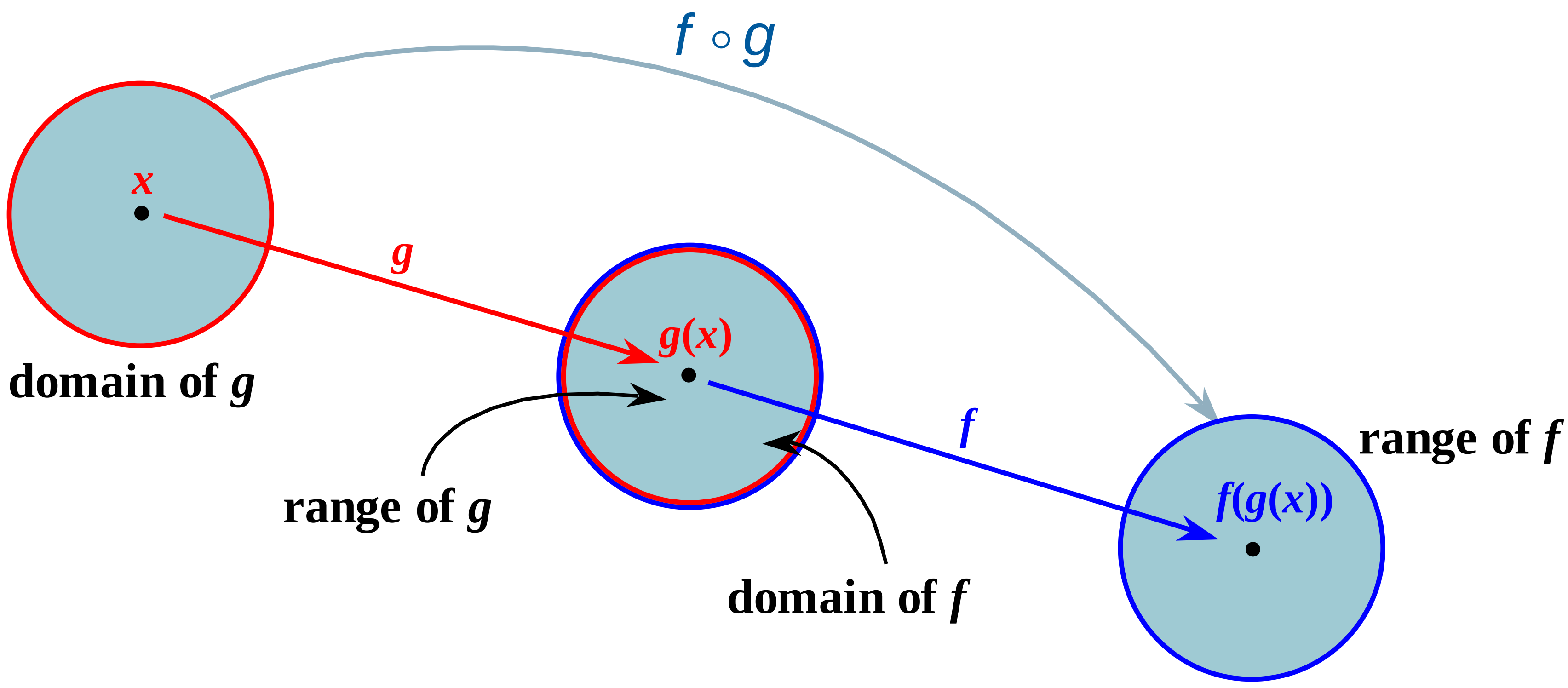
Composition of functions

- Composition of functions is the successive application of the functions in a specific order.

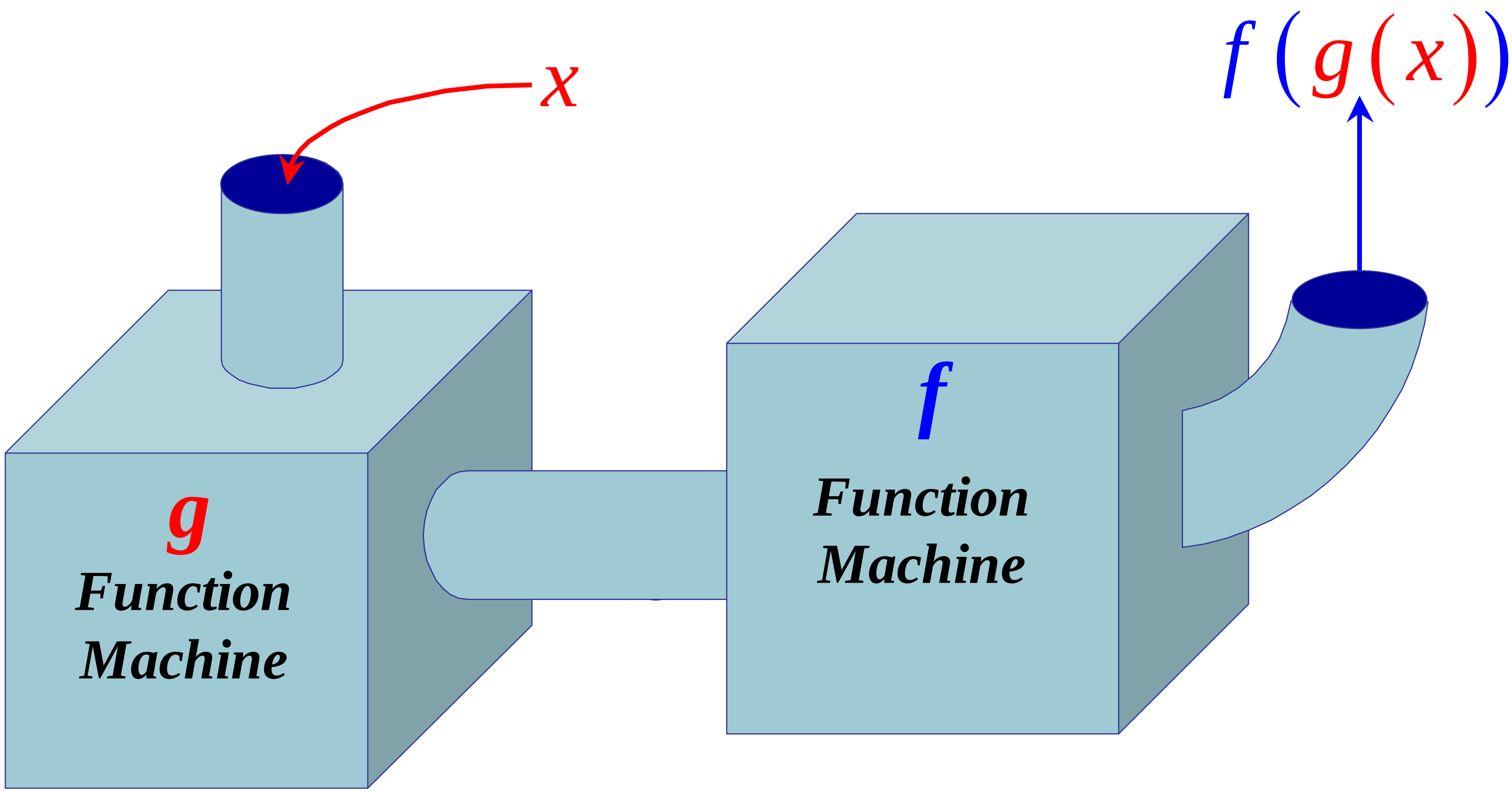
- Given two functions *f* and *g*, the **composite function** $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$ and is read "*f* of *g* of *x*."

- The domain of $f \circ g$ is the set of elements *x* in the domain of *g* such that *g*(*x*) is in the domain of *f*.
  - Another way to say that is to say that "the range of function *g* must be in the domain of function *f*."

A composite function



A different way to look at it…



Example 3

- Evaluate $(f \circ g)(x)$ and $(g \circ f)(x)$:
  - $f(x) = x - 3$
  - $g(x) = 2x^2 - 1$

$$f(g(x)) = (2x^2 - 1) - 3$$
$$= 2x^2 - 4$$

$$g(f(x)) = 2(x - 3)^2 - 1$$
$$= 2(x^2 - 6x + 9) - 1$$
$$= 2x^2 - 12x + 18 - 1$$

$$(f \circ g)(x) = 2x^2 - 4$$

$$(g \circ f)(x) = 2x^2 - 12x + 17$$

You can see that function composition is not commutative!

Example 4

- Find the domain of $(f \circ g)(x)$ and $(g \circ f)(x)$:
  - $f(x) = x - 1$
  - $g(x) = \sqrt{x}$

$$(f \circ g)(x) = \sqrt{x} - 1 \qquad D_{f \circ g} = \{x : x \geq 0\}$$

(Since a radicand can't be negative in the set of real numbers, *x* must be greater than or equal to zero.)

$$(g \circ f)(x) = \sqrt{x - 1} \qquad D_{g \circ f} = \{x : x \geq 1\}$$

(Since a radicand can't be negative in the set of real numbers, $x - 1$ must be greater than or equal to zero.)

Example 5

- The number of bicycle helmets produced in a factory each day is a function of the number of hours (t) the assembly line is in operation that day and is given by $n = P(t) = 75t - 2t^2$.
- The cost C of producing the helmets is a function of the number of helmets produced and is given by $C(n) = 7n + 1000$.
- Determine a function that gives the cost of producing the helmets in terms of the number of hours the assembly line is functioning on a given day.
- Find the cost of the bicycle helmets produced on a day when the assembly line was functioning 12 hours.
  (solution on next slide)

Solution to Example 5:

$n = P(t) = 75t - 2t^2$ $\quad\quad$ $C(n) = 7n + 1000$

- Determine a function that gives the cost of producing the helmets in terms of the number of hours the assembly line is functioning on a given day.
  - $$\begin{aligned}\text{Cost} = C(n) &= C(P(t)) \\ &= C(75t - 2t^2) \\ &= 7(75t - 2t^2) + 1000 \\ &= -14t^2 + 525t + 1000\end{aligned}$$
  - Find the cost of the bicycle helmets produced on a day when the assembly line was functioning 12 hours.

$$C = -14t^2 + 525t + 1000 = \$5284$$

Summary…

- Function arithmetic – add the functions (subtract, etc)
  - Addition
  - Subtraction
  - Multiplication
  - Division
- Function composition
  - Perform function in innermost parentheses first
  - Domain of “main” function must include range of “inner” function