T101 SECTION 1-2 EXPLORATIONS WITH PATTERNS

"*Solving a problem means finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable.*" *(George Polya)*

"...students should investigate numerical and geometric ***patterns*** *and express them mathematically in words or in symbols. They should analyze the structure of the* ***pattern*** *and how it grows or changes, organize this information systematically, and use their analysis to develop generalizations about the mathematical relationships in the* ***pattern****."* *(NCTM Principles and Standards, p. 159)*

I INTRODUCTION TO PATTERNS

Consider each of the following:

1, 3, 9, 27, 81, ...

How would you describe this pattern?

What number would be next?

1, 2, 4, ...

What number do you think comes next?

Could there be a different number?

☺ ☹ ☺ ☹ ☹ ☺ ☹ ☹ ☹...

Describe this pattern.

Note that patterns do not have to be numerical.

Describe the following pattern:

$1 + 0 \cdot 9 =$ ___________

$2 + 1 \cdot 9 =$ ___________

$3 + 12 \cdot 9 =$ ___________

$4 + 123 \cdot 9 =$ ___________

$5 + 1234 \cdot 9 =$ ___________

________________________ = _____________

________________________ = _____________

Does this pattern continue forever?
Try it.

As you can see, sometimes determining a pattern on the basis of only a few cases is not reliable.

II. INDUCTIVE REASONING

Inductive Reasoning: the method of making generalizations based on observations and patterns.

However, since one is only using collected data, it is sometimes difficult to know if all cases have been checked. Thus, inductive reasoning may lead to a **conjecture**, a statement thought to be true, but not yet proved true or false.

For instance, in the above example, had we stopped at 10 we may have used inductive reasoning to conclude that the product was ALWAYS "*n* ones, (where *n* is a natural number greater than 1)". But going one further to 11 provided the *counterexample* that we need to disprove this *conjecture.* It is easier to prove a conjecture is false than to prove it is true. To prove a conjecture false, you only need one ***counterexample, but the lack of a counterexample does not automatically make the conjecture true!***

III. SEQUENCES (Arithmetic, Geometric, and Others)

A sequence is an ordered arrangement of numbers, figures, or objects. Each of the numbers, figures, or objects is called a term and is identified as the 1st term, 2nd term, 3rd term, etc.

Sequence	Next 2 Terms	Pattern
1, 2, 3, 4, 5, ...		
7, 11, 15, 19, 23, ...		
2, 22, 222, 2222, ...		
1, 1, 2, 3, 5, 8, 13, …		
1, 2, 4, 7, 11, 16, ...		

What do the first two sequences have in common?

These first two sequences are called ______________________________ because each successive term is found by adding (or subtracting) a fixed value. The fixed value that is added or subtracted each time is called the ___________________ (meaning the difference between each successive term).

A. ARITHMETIC SEQUENCES

Let's look more closely at the sequence: 7, 11, 15, 19...

What is the 1st term? __________ This is generally denoted by : $\boldsymbol{a_1}$

What is the difference between each term? __________ This is generally denoted by : $\boldsymbol{d}$

To investigate, let us make a table:

n (Term #)	$\boldsymbol{a_n}$ Actual nth Term	How was this term found?	Write algebraically (symbolically)
1	a_1 (or 1st term) **is 7**	(Given) 7	a_1
2	a_2 (or 2nd term) **is 11**		
3	a_3 (or 3rd term) **is 15**		
4	a_4 (or 4th term) **is 19**		
5	a_5 (or 5th term) **is** ________		
10	a_{10} (or 10th term) **is** ________		
15	a_{15} (or 15th term) **is** ________		
100	a_{100} (or 100th term) **is** ________		
n	a_n (or nth term) **is** ______________		

You have just derived the formula for an ***arithmetic sequence with first term a_1 and difference of d:***

<u>FINDING A SPECIFIC TERM OF AN ARITHMETIC SEQUENCE</u>

An arithmetic sequence has a 1st term of 2 and 55th term of 164, find the 100th term.

APPLICATIONS

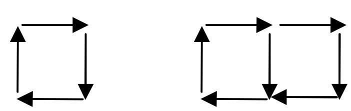
Given the following sequence: 10 + 14 + 18 + 22 + . . . + 326

a. What type of sequence is this?

b. How many terms are in this sequence?

c. Find the sum.

Matchstick Problem: Ryan is building matchstick squares sequences (see below). He uses 67 matchsticks to form the **last** figure in his sequence. How many matchsticks does he use for the entire project?

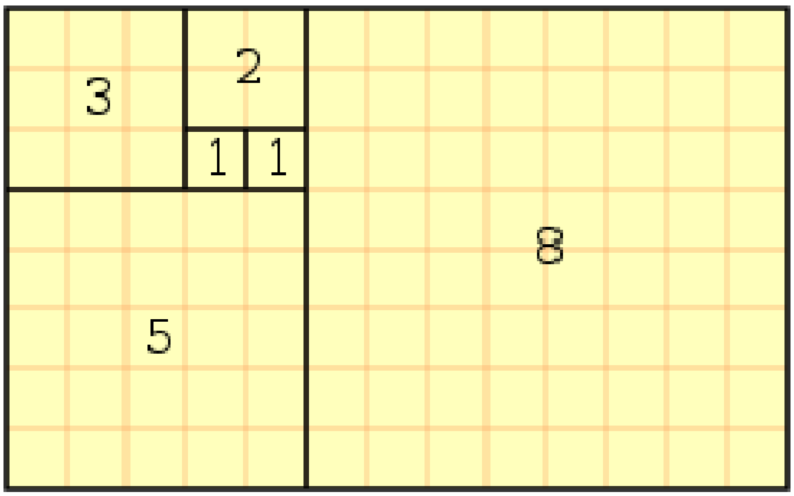


B. THE FIBONACCI SEQUENCE

The sequence is named after the Italian Leonardo de Pisa, better known by his nickname Fibonacci. Each term is the sum of the two terms that precede it.

Thus: $F_n = (F_{n-1}) + (F_{n-2})$

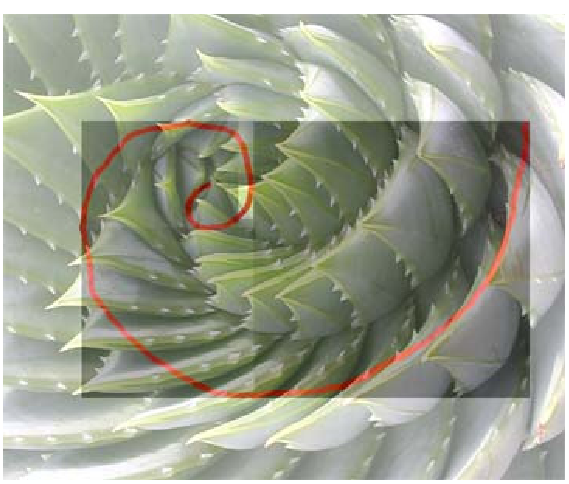
0, 1, 1, 2, 3, 5, 8, 13, _____ , _____ , _____ , _____ , _____ , …



The Fibonacci Spiral:
is a geometric spiral whose growth is regulated by the Fibonacci Series. Its sudden, almost exponential growth parallels the rapid growth of the series itself.

The spiral itself is a series of connected quarter-circles drawn inside an array of squares with Fibonacci numbers for dimensions.

Fibonacci in Nature:
Now, it just so happens that we see such a Fibonacci Spiral repeated time and time again in nature. In fact, while symmetry is often seen in our world (for example, we have two symmetrically placed arms, legs, eyes, ears, nostrils, etc.), there are *actually more examples of the asymmetry of the Fibonacci Series*.



C. Geometric Sequences

In a geometric sequence, each consecutive term is found from its predecessor by ***multiplying*** by a fixed number, called the ***ratio (r)***.

EXAMPLE: Two bacteria are in a dish. The number of bacteria triples every hour. Find the number of bacteria after 10 hours (beginning of hour 11) and after n hours.

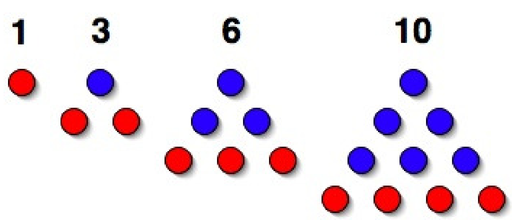
n (Term #)	a_n Actual nth Term	How was this term found?	Write algebraically (symbolically)
1 Beginning of Hour 1	a_1 (or 1st term) **is 2**	(Given) 2	a_1
2	a_2 (or 2nd term) **is** ________		
3	a_3 (or 3rd term) **is** ________		
4	a_4 (or 4th term) **is** ________		
8	a_5 (or 8th term) **is** ________		
n	a_n (or nth term) **is** ______________		

You have just derived the formula for a ***geometric sequence with first term a_1 and ratio of r, ($r \neq 0$):***

D. Other Sequences

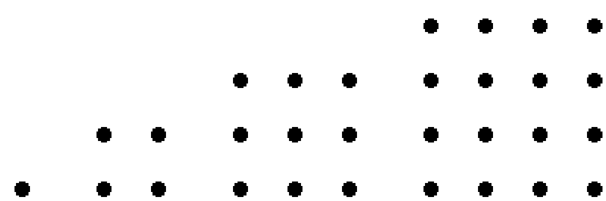
Figurate Numbers give us examples of sequences that are neither arithmetic nor geometric. They can be represented by dots arranged in the shape of certain geometric figures.

Triangular Numbers



n	*# of dots*	*Pattern?*
1	1	1
2	3	1 + 2
3	6	1 + 2 + 3
4	10	1 + 2 + 3 + 4
5		
15		
n		

Square Numbers



n	*# of dots*	*Pattern?*
1	1	
2	4	
3		
4		
5		
20		
n		

Successive Differences

Sometimes the sequence is not arithmetic, geometric, or any other obvious pattern. Another method to determine the pattern within a sequence is to examine the difference between the differences. For instance, find the 7th and 8th term in each of the following sequences:

5 6 14 29 51 80 ______ ______

(First Difference) 1 8

(Second Difference) 7

Example: Find the first four terms of a sequence whose n*th* term is given by the following formula, then determine if the sequence is arithmetic, geometric or neither.

$a_n = 6n + 7$

$a_1 =$ ________

$a_2 =$ ________

$a_3 =$ ________

$a_4 =$ ________

$a_n = n^2 - 2$

$a_1 =$ ________

$a_2 =$ ________

$a_3 =$ ________

$a_4 =$ ________

TAKE HOME PROBLEMS

An arithmetic sequence has a 1st term of 2 and 5th term of 26, find the 100th term.

How many numbers are in the sequence 3, 9, 15, 21, 27, ..., 159? Then find the sum of the sequence.

Find the next two terms using successive differences:

2 3 9 23 48 87 _______ ________

(First Difference)

(Second Difference)

(Third Difference)

Find the first four terms of a sequence whose n*th* term is given by the following formula.

$a_n = -2n^3 - 5$

$a_1 =$ _________

$a_2 =$ _________

$a_3 =$ _________

$a_4 =$ _________