The Continued Fraction for e
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I have shown the continued fraction for e to students (in number theory classes) for years, and I was finally motivated to learn the proof of this famous and beautiful formula. I found a clear outline of the proof in a book written by a professor of mine at Reed College: Joe Roberts, *Elementary Number Theory: A Problem Oriented Approach*, MIT Press, Cambridge, Massachusetts, 1977. I should mention a charming feature of the book: it was entirely caligraphed by hand.

The following is a sketch of my solution to problem 22 of chapter XIII (p. 131). Roberts credits the solution outlined in that problem to Euler, Hurwitz, and the Reed professor (and Apple Distinguished Scientist) Richard E. Crandall. (I'm not certain how much of this is due to the latter, however; the problem this is a solution of continues beyond what I've shown here.)

Recall a simple continued fraction is any expression of the form

$$[a_0, a_1, a_2, a_3, a_4, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

where usually (but not always) the partial denominators a_i are taken to be positive integers.

Here, we will show that e, the base of the natural logarithms, is given by

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$$

We will also show that

$$\frac{e^2+1}{e^2-1} = [1, 3, 5, 7, 9, \ldots]$$

and more generally that

$$\frac{e^{2x}+1}{e^{2x}-1} = \left[\tfrac{1}{x}, \tfrac{3}{x}, \tfrac{5}{x}, \tfrac{7}{x}, \ldots\right] \text{ for } x \neq 0.$$

Now we can define the kth convergent to be

$$C_k = [a_0, a_1, a_2, \ldots, a_k] = \frac{p_k}{q_k}$$

and we define the value of the continued fraction to be the limit $\lim_{k\to\infty} C_k$ if the limit exists. If the partial denominators are all positive integers, the continued fraction does indeed converge.

It turns out that the convergents are easy to compute recursively using the following formulas:

$$\begin{aligned} p_0 &= a_0, & q_0 &= 1 \\ p_1 &= a_0 a_1 + 1, & q_1 &= a_1 \end{aligned}$$

and for $k \geq 2$,

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \\ q_k &= a_k q_{k-1} + q_{k-2} \end{aligned}$$

These can be proved by induction.

We now outline the proof of the continued fraction formulas promised above. We begin with

$$\frac{e^{2x}+1}{e^{2x}-1} = \left[\tfrac{1}{x}, \tfrac{3}{x}, \tfrac{5}{x}, \tfrac{7}{x}, \ldots\right].$$

Define a function $f_n(x)$ by $\displaystyle\sum_{s=0}^{\infty} a_{ns} x^{2s}$, where $a_{ns} = \dfrac{(n+s)!}{s!(2n+2s)!}$, when the series converges. It turns out this converges for all x (by the ratio test).

Now with a bit of algebra with the coefficients, it is not difficult to show that

$$f_n(x) - (4n+2) f_{n+1}(x) = 4x^2 f_{n+2}(x)$$

for $x \geq 0$.

It is also easy to verify that

$$\frac{f_0(x)}{f_1(x)} = 2x \frac{e^{2x}+1}{e^{2x}-1}.$$

This is because $f_0(x) = \sum_{s=0}^{\infty} x^{2s}/(2s)! = \cosh x = \frac{1}{2}(e^x + e^{-x})$; while $f_1(x)$ is almost $\sinh x = \frac{1}{2}(e^x - e^{-x})$ (that's where the $2x$ comes from).

So now we can develop the continued fraction:

$$
\begin{aligned}
\frac{e^{2x}+1}{e^{2x}-1} &= \frac{1}{2x}\frac{f_0(x)}{f_1(x)} \\
&= \frac{1}{2x}\cdot\frac{2f_1(x)+4x^2 f_x(x)}{f_1(x)} \\
&= \frac{1}{x}+\frac{2xf_2(x)}{f_1(x)} \\
&= \frac{1}{x}+\frac{2xf_2(x)}{6f_2(x)+4x^2 f_3(x)} \\
&= \frac{1}{x}+\cfrac{1}{\frac{3}{x}+\cfrac{2xf_3(x)}{f_2(x)}} \\
&= \frac{1}{x}+\cfrac{1}{\frac{3}{x}+\cfrac{2xf_3(x)}{10f_3(x)+4x^2 f_4(x)}} \\
&= \frac{1}{x}+\cfrac{1}{\frac{3}{x}+\cfrac{2xf_3(x)}{10f_3(x)+4x^2 f_4(x)}} \\
&= \frac{1}{x}+\cfrac{1}{\frac{3}{x}+\cfrac{1}{\frac{5}{x}+\cfrac{2xf_4(x)}{f_3(x)}}} \\
&\vdots
\end{aligned}
$$

This process continues by induction, giving us

$$\frac{e^{2x}+1}{e^{2x}-1} = \left[\tfrac{1}{x}, \tfrac{3}{x}, \tfrac{5}{x}, \tfrac{7}{x}, \ldots\right].$$

With $x = 1$, we obtain

$$\frac{e^2+1}{e^2-1} = [1, 3, 5, 7, 9, \ldots].$$

With $x = 1/2$, we obtain

$$\frac{e+1}{e-1} = [2, 6, 10, 14, 18, \ldots, 4n+2, \ldots].$$

We will now pursue the continued fraction for e. This is developed from the last continued fraction above, but it's rather tricky.

Define $\alpha = [a_0, a_1, \ldots]$, where

$$\begin{aligned} a_0 &= 2, \\ a_{3n} &= a_{3n+1} = 1, \\ a_{3n-1} &= 2n. \end{aligned}$$

So $\alpha = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$. We wish to show that $\alpha = e$. Let the convergents of α be $\dfrac{p_n}{q_n}$, $n \geq 0$. As above, we know that

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2} \end{aligned}$$

for $n \geq 2$. This allows us to write, using the definition of the numbers a_n

$$\begin{aligned} p_{3n+1} &= p_{3n} + p_{3n-1} \\ p_{3n} &= p_{3n-1} + p_{3n-2} \\ p_{3n-1} &= 2n p_{3n-2} + p_{3n-3} \\ p_{3n-2} &= p_{3n-3} + p_{3n-4} \\ p_{3n-3} &= p_{3n-4} + p_{3n-5} \end{aligned}$$

A suitable linear combination of these produces the following equation (valid for $n \geq 2$):

$$p_{3n+1} = (4n+2)p_{3n-2} + p_{3n-5} \text{ for } n \geq 2.$$

In an identical manner, we can prove

$$q_{3n+1} = (4n+2)q_{3n-2} + q_{3n-5} \text{ for } n \geq 1.$$

Now if we let $c_n = p_{3n+1}$ and $d_n = q_{3n+1}$, we see that these obey

$$\begin{aligned} c_n &= (4n+2)c_{n-1} + c_{n-2} \\ d_n &= (4n+2)d_{n-1} + d_{n-2} \end{aligned}$$

The point of this is that these mirror the convergents for the continued fraction

$$\frac{e+1}{e-1} = [2, 6, 10, 14, 18, \ldots, 4n+2, \ldots].$$

Indeed, if we let the convergents for this continued fraction be P_n/Q_n, we have

$$\begin{aligned} P_n &= (4n+2)P_{n-1} + P_{n-2} \\ Q_n &= (4n+2)Q_{n-1} + Q_{n-2} \end{aligned}$$

This suggests that $c_n = P_n$ and $d_n = Q_n$, but in fact, that's not quite true. Instead, it turns out that

$$P_n = \tfrac{1}{2}(c_n + d_n)$$
$$Q_n = \tfrac{1}{2}(c_n - d_n)$$

This is easy to verify, because it is true for the first two terms:

n	c_n	d_n	P_n	Q_n
0	3	1	2	1
1	19	7	13	6

For example, if $n = 1$, we have $P_1 = 13 = \frac{1}{2}(c_1 + d_1) = \frac{1}{2}(19 + 7)$. Then, for $n \geq 2$, it follows by induction. (Note linear combinations of c_n and d_n will follow the same recursion as do c_n and d_n.)

We are now in a position to prove the continued fraction for e. We have just verified that

$$P_n = \tfrac{1}{2}(c_n + d_n)$$
$$Q_n = \tfrac{1}{2}(c_n - d_n)$$

which is actually

$$P_n = \tfrac{1}{2}(p_{3n+1} + q_{3n+1})$$
$$Q_n = \tfrac{1}{2}(p_{3n+1} - q_{3n+1})$$

So now we have

$$\frac{P_n}{Q_n} = \frac{\frac{1}{2}(p_{3n+1} + q_{3n+1})}{\frac{1}{2}(p_{3n+1} - q_{3n+1})} = \frac{\dfrac{p_{3n+1}}{q_{3n+1}} + 1}{\dfrac{p_{3n+1}}{q_{3n+1}} - 1}$$

In the limit as $n \to \infty$, this becomes

$$\frac{e+1}{e-1} = \frac{\alpha + 1}{\alpha - 1},$$

and since $e > 1$ and $\alpha > 1$, it follows that $\alpha = e$.