15.2. Continued fractions for rational numbers

Here's an example: Perform the Euclidean algorithm on 47 and 17.

$$\begin{aligned}47 &= 2(17) + 13\\ 17 &= 1(13) + 4\\ 13 &= 3(4) + 1\end{aligned}$$

Now write these equations in the form

$$\begin{aligned}\frac{47}{17} &= 2 + \frac{13}{17}\\ \frac{17}{13} &= 1 + \frac{4}{13}\\ \frac{13}{4} &= 3 + \frac{1}{4}\end{aligned}$$

Now substitute:

$$\begin{aligned}\frac{47}{17} = 2 + \frac{13}{17} &= 2 + \cfrac{1}{\frac{17}{13}}\\ &= 2 + \cfrac{1}{1 + \frac{4}{13}}\\ &= 2 + \cfrac{1}{1 + \cfrac{1}{\frac{13}{4}}}\\ &= 2 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{4}}}\end{aligned}$$

In this way, we can write any positive rational number a/b as a **finite simple continued fraction**

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots + \cfrac{1}{a_n}}}}}$$

(This is Theorem 15.1 in Burton.) We abbreviate this expression as $[a_0; a_1, a_2, a_3, a_4, \ldots, a_n]$. We refer to the numbers a_i as partial denominators; usually but not always, they are integers. We define the kth convergent by

$$C_k = [a_0; a_1, a_2, \ldots, a_k] = \frac{p_k}{q_k}$$

(so p_k and q_k are integers, the numerator and denominator of the convergent written as an ordinary fraction).

If you happen to know the partial denominators $a_0, a_1, a_2, \ldots$, you can quickly compute the convergents: Let

$$\begin{aligned} p_0 &= a_0, & q_0 &= 1 \\ p_1 &= a_0 a_1 + 1, & q_1 &= a_1 \end{aligned}$$

and for $k \geq 2$ let

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \\ q_k &= a_k q_{k-1} + q_{k-2} \end{aligned}$$

The proof of this uses induction. We therefore assume the statement is true for $n = k$ and try to verify it for $n = k + 1$. So we are assuming that

$$[a_0; a_1, a_2, \ldots, a_k] = \frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}$$

Now usually, the partial denominators a_i are supposed to be integers. But we don't have to assume that here. So by the induction hypothesis, this is supposed to be true whatever the kth partial denominator is. So it will be true if we replace a_k by $a_{k+1} + 1/a_k$:

$$[a_0; a_1, a_2, \ldots, a_{k+1} + \tfrac{1}{a_k}] = \frac{p_k}{q_k} = \frac{\left(a_{k+1} + \frac{1}{a_k}\right) p_{k-1} + p_{k-2}}{\left(a_{k+1} + \frac{1}{a_k}\right) q_{k-1} + q_{k-2}}$$

But with a little algebra, this is

$$\begin{aligned}[a_0; a_1, a_2, \ldots, a_k, a_{k+1}] &= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\ &= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}}\end{aligned}$$

so we are done. (We still need to verify that the statement is true for $n = 1$, but that's easy to do.)

Here's an example as to how to use the algorithm to evaluate a continued fraction. We will try $[3; 4, 5, 6, 7]$. We get

$$\begin{aligned}p_0 &= a_0 = 3 \\ q_0 &= 1 \\ p_1 &= a_0 a_1 + 1 = 3(4) + 1 = 13 \\ q_1 &= a_1 = 4 \\ p_2 &= a_2 p_1 + p_0 = 5(13) + 3 = 68 \\ q_2 &= a_2 q_1 + q_0 = 5(4) + 1 = 21 \\ p_3 &= a_3 p_2 + p_1 = 6(68) + 13 = 421 \\ q_3 &= a_3 q_2 + q_1 = 6(21) + 4 = 130 \\ p_4 &= a_4 p_3 + p_2 = 7(421) + 68 = 3015 \\ q_4 &= a_4 q_3 + q_2 = 7(130) + 21 = 931\end{aligned}$$

which tells us that

$$[3; 4, 5, 6, 7] = C_4 = \frac{p_4}{q_4} = \frac{3015}{931} = 3.23845\cdots$$

Of course, we can always simplify directly by hand:

$$[3; 4, 5, 6, 7] = 3 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{6 + \cfrac{1}{7}}}} = \frac{3015}{931}.$$

Why is all of this interesting? There are various reasons, but one reason is given by the theorem that for $1 \leq k \leq n$,

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}.$$

(This is easy to prove by induction, using the recursion for p_k and q_k.) This gives as a nice way of solving linear Diophantine equations.

Example: Solve $47x - 17y = 1$. The solution is to find the convergents of the continued fraction for 47/17. We arrive at $p_1 = 11$, $q_2 = 4$ and $p_3 = 47$, $q_3 = 17$. So

$$p_3q_2 - q_3p_2 = (-1)^{3-1} = 1$$

or

$$47(4) - 17(11) = 1$$

So we take $x_0 = 4$ and $y_0 = 11$ and build the rest of the solution from there as usual. (That would be $x = 4 + 17t$ and $y = 11 + 47t$.)

15.3. Infinite continued fractions.

In this section, we consider unending expressions of the form

$$[a_0; a_1, a_2, a_3, a_4, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

If we define the convergents C_k as before, it can be proved (Theorem 15.4) that

$$C_0 < C_2 < C_4 < \ldots < C_5 < C_3 < C_1$$

and from $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$ or

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}},$$

it follows $C_k - C_{k-1} \to 0$ (since the partial quotients q_k increase). This forces the limit $\lim_{k\to\infty} C_k$ to exist. So we can safely define

$$[a_0; a_1, a_2, a_3, a_4, \ldots] = \lim_{n\to\infty} C_n = \lim_{n\to\infty} [a_0; a_1, a_2, a_3, \ldots, a_n].$$

Now from the last section we realize that a finite continued fraction always represents a rational number, and every positive rational number has a finite continued fraction representation. So if $a_k > 0$ for all $k \geq 1$, the infinite continued fraction will represent an irrational number.

Here's a simple example: Let $x = [1; 1, 1, 1, \ldots]$. This is:

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = 1 + \frac{1}{x}$$

We can solve $x = 1 + 1/x$ easily to obtain $x = (1 + \sqrt{5})/2$, a number known as the "golden ratio." If you set up the recursion to find the numerators and denominators of the convergents, you will quickly discover they are all Fibonacci numbers. (Recall the Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$; where each number is the sum of the preceding two numbers.)

We already know that every infinite continued fraction represents an irrational number (if the partial denominators are all positive). But it turns out that every positive real number has a continued fraction expansion. Here is how to find that expansion. Suppose $x > 0$ is represented as a continued fraction, so

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = a_0 + \frac{1}{\text{something bigger than 1}}.$$

So we must have $a_0 = [x]$, where $[x]$ is the largest integer less than or equal to x. (E.g., $[\pi] = 3$. The greatest integer function is often implemented in computer programming languages as the "floor" function.) Now

$$x - [x] = x - a_0 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

so we have

$$a_1 + \frac{1}{a_2 + \cdots} = \frac{1}{x - a_0}$$

and therefore,

$$a_1 = \left[\frac{1}{x - a_0}\right]$$

This process continues. We have the following simple algorithm: Let $x_0 = x$. Then for $k \geq 0$,

$$a_k = [x_k] \quad \text{and} \quad x_{k+1} = \frac{1}{x_k - a_k}.$$

Here's an example: $x = \sqrt{3}$. We have $x_0 = \sqrt{3}$ and $a_0 = [x_0] = [\sqrt{3}\,] = 1$. Then we have

$$x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2}.$$

So $a_1 = [x_1] = \left[(\sqrt{3} + 1)/2\right] = 1$. Then we have

$$x_2 = \frac{1}{x_1 - a_1} = \cfrac{1}{\cfrac{\sqrt{3} + 1}{2} - 1} = \frac{2}{\sqrt{3} - 1} = \sqrt{3} + 1.$$

So $a_2 = [x_2] = \left[\sqrt{3} + 1\right] = 2$. Next we have

$$x_3 = \frac{1}{x_2 - a_2} = \frac{1}{\sqrt{3} + 1 - 2} = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2}.$$

So $a_3 = [x_3] = 1$, but $x_3 = x_1$, so we realize that the calculation repeats! It follows that

$$\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \ldots] = [1; \overline{1, 2}\,].$$

It turns out that every repeating infinite continued fraction represents a quadratic irrationality (a solution of a quadratic equation with integer coefficients), and every quadratic irrationality has a repeating continued fraction. That is, it has a continued fraction of the form

$$[a_0; a_1, \ldots, a_k, \overline{b_{k+1}, \ldots, b_{k+n}}\,]$$

Why is all of this interesting? It turns out that continued fraction convergents are the best possible rational approximations to a number x, in the sense of the following theorem: Suppose $C_n = p_n/q_n$ is a convergent for x and suppose $1 \le b \le q_n$. Then

$$\left| x - \frac{p_n}{q_n} \right| \le \left| x - \frac{a}{b} \right|$$

That is, of all fractions with denominator no bigger than q_n, the fraction p_n/q_n approximates x the best. We can always approximate an irrational number x more closely by choosing a fraction a/b with a large denominator. For example, we can approximate π very closely, to 6 decimals, by using 3141592/1000000. But it happens that $\pi = [3; 7, 15, 1, 292, \ldots]$ and the third convergent is $355/133 = [3; 7, 15, 1] = 3.14159292\cdots$ while $\pi = 3.14159265\cdots$. So we can achieve a 6 decimal approximation using a fraction with a denominator as small as 113 instead of a denominator with 7 digits. The unexpected closeness of this approximation is because the next partial denominator, $a_4 = 292$, is large. By the way, another convergent for π is $C_1 = 22/7$, a familiar approximation known in antiquity.

Here is another application of continued fractions: They can be used to solve Pell's equation, the nonlinear Diophantine equation $x^2 - dy^2 = 1$ (where $d > 0$ is an integer). It turns out that if $x = p$ and $y = q$ are positive integers solving the equation, then p/q is a convergent for $\sqrt{d}$. Burton proves this, and gives a lot more information about the solutions of this equation, in section 15.4.

We conclude with several pretty formulas. The continued fraction for π appears to have no simple pattern. But in 1737 (according to Burton) Euler proved that

$$\frac{e^2 + 1}{e^2 - 1} = [1; 3, 5, 7, 9, \ldots]$$

and also that

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$$

There is a pretty non-simple continued fraction expansion for π:

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}$$

Burton says this was due to William Brouncker and dates to 1655.