2.2 and 2.3. More About Determinants

The determinant function has several simple properties, which can be used to efficiently find determinants using row operations. Let A be an $n \times n$ matrix. Then:

1. If A has a row of all zeros, or a column of all zeros, then $\det A = 0$. Also, if A has two identical rows, or two identical columns, then $\det A = 0$.
2. $\det(A^T) = \det A$.
3. If B is the matrix obtained by multiplying one row or one column of A by the scalar k, then $\det B = k \det A$.
4. If B is the matrix obtained by interchanging two rows (or two columns) of A, then $\det B = -\det A$.
5. If B is the matrix obtained by adding a multiple of one row of A to another row of A, or by adding a multiple of one column to another column, then $\det B = \det A$.
6. Suppose A, B and C are $n \times n$ matrices that are exactly the same, except for row i. Suppose row i of C is the sum of row i of A and row i of B. Then $\det C = \det A + \det B$.

The book doesn't give proofs of these statements. But they give several 3×3 examples to show why they work. For example, why would a matrix with two identical rows have determinant zero? Say A is a 3×3 matrix with its second and third rows equal. Say

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Then we have

$$\begin{aligned} \det A &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{23} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{21} & a_{23} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}\left(a_{22}a_{23} - a_{23}a_{22}\right) - a_{12}\left(a_{21}a_{23} - a_{23}a_{21}\right) + a_{13}\left(a_{21}a_{22} - a_{22}a_{21}\right) \\ &= 0 - 0 + 0 = 0. \end{aligned}$$

But what about a 4×4 matrix with two equal rows? The determinant calculation would involve four 3×3 determinants, each with a repeated row. So that would end up being zero too.

Here's another example. Why would the determinant not change if you add a multiple of one row to another? Say A is any 3×3 matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and say B is obtained by adding k times row two to row one in A. So

$$B = \begin{bmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We show $\det B = \det A$ by expanding along the first row:

$$\begin{aligned}
\det B &= (a_{11}+ka_{21}) \begin{vmatrix} a_{22} & a_{33} \\ a_{23} & a_{32} \end{vmatrix} - (a_{12}+ka_{22}) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (a_{13}+ka_{23}) \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{33} \\ a_{23} & a_{32} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} \\
&\quad + k \left(a_{21} \begin{vmatrix} a_{22} & a_{33} \\ a_{23} & a_{32} \end{vmatrix} - a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} \right) \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= \det A \quad + \quad 0 \\
&= \det A.
\end{aligned}$$

(The last 3×3 determinant is zero because it has a repeated row.)

These properties enable us to find determinants of matrices by using row reduction. The trick is to row reduce until the matrix is of upper triangular form, because it turns out that the determinant of an upper triangular matrix is exactly the product of the entries on the diagonal. This is easy to see, if you try a 3×3

example, say:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} 0 & a_{22} \\ 0 & 0 \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} - 0 + 0 = a_{11}a_{22}a_{33}.$$

(The latter two 2×2 determinants are zero because they have columns of zeros.) Something similar will happen if you try a 4×4 example.

So to find determinants, proceed as follows: Perform row operations to reduce the matrix to upper triangular form. For each row operation, track how it changes the value of the determinant according to properties 3, 4 and 5 of the list above.

Example.

$$\begin{vmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 4 & 0 \\ -1 & 5 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -2 & -5 \\ 0 & 1 & 3 & -3 \\ 0 & 7 & 3 & 6 \end{vmatrix} \quad \text{(property 5)}$$
$$= (-1)\begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & -3 \\ 0 & -3 & -2 & -5 \\ 0 & 7 & 3 & 6 \end{vmatrix} \quad \text{(property 4)}$$
$$= (-1)\begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 7 & -14 \\ 0 & 0 & -18 & 27 \end{vmatrix} \quad \text{(property 5)}$$
$$= (-1)(7)\begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -18 & 27 \end{vmatrix} \quad \text{(property 3)}$$
$$= (-1)(7)\begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -9 \end{vmatrix} \quad \text{(property 5)}$$
$$= (-1)(7)(1)(1)(1)(-9) = 63.$$

Determinants of products of matrices. Now we consider an important property of matrices: $\det(AB) = \det(A)\det(B)$ (for $n \times n$ matrices). This is not at all obvious, but it turns out we can see why it is true by using elementary matrices.

The key is to see that it is true for products EB, where E is an elementary matrix. But it turns out to be very easy to verify that $\det(EB) = \det(E)\det(B)$, using properties 3, 4 or 5 of the list above.

Now if A is invertible, we can write it as a product of elementary matrices, say

$$A = E_1 E_2 E_3 \cdots E_p.$$

(This can be seen by considering the process of finding A^{-1} by row-reducing A to the identity matrix.) Then

$$AB = E_1 E_2 E_3 \cdots E_p B.$$

Now we can take determinants and factor the right-hand side, using $\det(EB) = \det(E)\det(B)$. We obtain

$$\begin{aligned}\det(AB) &= \det(E_1)\det(E_2)\det(E_3)\cdots\det(E_p)\det(B)\\ &= \det(E_1 E_2 E_3 \cdots E_p)\det(B)\\ &= \det(A)\det(B)\end{aligned}$$

Now what if A is *not* invertible? Then it turns out you can row-reduce A to obtain a matrix with at least one row of all zeros along the bottom. (That's what happens when you cannot row-reduce to make it be I.) So that matrix has determinant 0. If A is not invertible, then AB cannot be invertible either. (If $(AB)^{-1} = C$ existed, then $B^{-1}A^{-1} = C$ so $A^{-1} = BC$ but A^{-1} doesn't exist.) So in that case, we still have $\det(AB) = \det(A)\det(B)$. In fact, $\det(AB) = \det(A)\det(B) = 0$.

One interesting consequence of the property $\det(AB) = \det(A)\det(B)$:

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

This is true because $A^{-1}A = I$, so $\det(A^{-1}A) = \det(I)$ or $\det(A^{-1})\det(A) = 1$. One particular and important point: This means A is invertible exactly when $\det A \neq 0$.

Now we update a theorem giving equivalent conditions for a matrix A:

Theorem. For an $n \times n$ matrix A, the following are equivalent:

(a) A is invertible.

(b) $AX = B$ has a unique solution X for any B.

(c) $AX = 0$ has the unique solution $X = 0$.

(d) The reduced row-echelon form of A is I.

(e) A is a product of elementary matrices.

(f) $\det A \neq 0$.

Another approach to determinants. Some texts define the determinant to be the real-valued function defined on $n \times n$ matrices that has the following properties:

1. If B is the matrix obtained by multiplying one row of A by the scalar k, then $\det B = k \det A$.
2. If B is the matrix obtained by interchanging two rows of A, then $\det B = -\det A$.
3. If B is the matrix obtained by adding a multiple of one row of A to another row of A, then $\det B = \det A$.
4. $\det(I) = 1$ (where I is the $n \times n$ identity matrix).

Then, with this definition, one verifies that determinants can be computed recursively by expanding along rows in terms of cofactors. This also establishes that the determinant function, defined in this way, really exists (and is unique). So this approach is the reverse of what we (and our textbook) actually did.