1.1. Linear Systems and Gaussian Elimination.

A linear equation in variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are just numbers (constants). Examples:

$$3x + 4y + 5z = 0$$

$$x - y + \tfrac{1}{2}z = \tfrac{3}{2}$$

$$x_1 + x_2 + 2x_3 + 3x_4 = 5$$

but the following are not linear:

$$\frac{2x}{x-y} = y$$

$$y = \sin x$$

$$x_1^2 + x_2^2 + x_3^2 = 1$$

The terminology "linear" comes from the fact that the graph of a linear equation in two variables, such as $3x + 4y = 12$, is a line.

We are interested in solving systems of linear equations.

Example. Solve

$$\begin{array}{rcrcr} 3x & - & 2y & = & 6 \\ x & - & y & = & 1 \end{array}$$

If we graph these two lines, we see they cross at $(4, 3)$, so $x = 4$, $y = 3$ is the solution. Algebraically, we can solve by substitution: Solve the second equation for y to get $y = x - 1$, then put into the first equation: $3x - 2y = 6$ becomes $3x - 2(x - 1) = 6$ so $3x - 2x + 2 = 6$ so $x = 4$. Then $y = x - 1 = 4 - 1 = 3$.

A different strategy for solving this same equation: Add a multiple of one equation to the other, to cancel out a variable. If we multiply the second equation by -2, we have the system:

$$\begin{array}{rcrcr} 3x & - & 2y & = & 6 \\ -2x & + & 2y & = & 2 \end{array}$$

Then if we add the two equations together, we get $x = 4$. Put that into either of the original equations to get $y = 3$.

Example.

$$\begin{array}{rcrcl} 2x & + & y & = & 4 \\ 4x & + & 2y & = & 3 \end{array}$$

These two lines turn out to parallel. They do not intersect, so there are no solutions to the system. The system is said to be inconsistent. Another example:

$$\begin{array}{rcrcl} 2x & + & y & = & 4 \\ 4x & + & 2y & = & 8 \end{array}$$

These two equations turn out to be exactly the same line. The system has infinitely many solutions. We might express the solution as $y = 4 - 2x$, so that whatever value we give the "free" or "independent" variable x, we then know what y is. Put another way, we can let x be any number, say t. Then $y = 4 - 2t$. That is, we can express the solution as

$$\begin{aligned} x &= t \\ y &= 4 - 2t \end{aligned}$$

The solution set is said to be parameterized by t. We say that this kind of system is dependent.

Example. Consider the following system:

$$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & 10 \\ & & 2y & + & 3z & = & -2 \\ & & & & 5z & = & -10 \end{array}$$

This can be solved by **back substitution**. Find z first: $z = -2$. Then find y. We have $2y + 3(-2) = -2$ so $y = 2$. Finally, find x. We have $2x + 3(2) - (-2) = 10$ so $x = 1$.

Example Consider the following system:

$$\begin{array}{rcrcrcr} & & y & + & 5z & = & 4 \\ x & - & 2y & + & 3z & = & 7 \\ 2x & & & + & z & = & 5 \end{array}$$

We can add multiples of equations together to put this into the nice "triangular" form where back-substitution works. Actually, swap the first two equations first:

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 7 \\ & & y & + & 5z & = & 4 \\ 2x & & & + & z & = & 5 \end{array}$$

Now add -2 times the first equation to the last equation:

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 7 \\ & & y & + & 5z & = & 4 \\ & & 4y & - & 5z & = & -9 \end{array}$$

Now add -4 times the second equation to the last equation:

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 7 \\ & & y & + & 5z & = & 4 \\ & & & & -25z & = & -25 \end{array}$$

This can be solved by back-substitution to obtain $z = 1$, $y = -1$ and $x = 2$.

Matrices. This last example can be done more easily by writing the equations as "augmented matrices". Then, instead of adding a multiple of an equation to another equation, we add a multiple of a row to another row. It looks like this: We start with

$$\begin{bmatrix} 0 & 1 & 5 & 4 \\ 1 & -2 & 3 & 7 \\ 2 & 0 & 1 & 5 \end{bmatrix}$$

Swap the first and second row to get:

$$\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & 5 & 4 \\ 2 & 0 & 1 & 5 \end{bmatrix}$$

Add -2 times the first row to the third row to get:

$$\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & 5 & 4 \\ 0 & 4 & -5 & -9 \end{bmatrix}$$

Add -4 times the second row to the third row to get:

$$\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -25 & -25 \end{bmatrix}$$

This matrix is said to be in row-echelon form ("echelon" means staircase). This is equivalent to the system

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 7 \\ & & y & + & 5z & = & 4 \\ & & & & -25z & = & -25 \end{array}$$

which we solve by back-substitution as before.

Example. Instead of reducing the matrix to triangular row-echelon form, we can do more row operations to reduce it to "reduced row-echelon form." So in the last example, we have

$$\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -25 & -25 \end{bmatrix}$$

but now we add 2 times the second row to the first row to get

$$\begin{bmatrix} 1 & 0 & 13 & 15 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -25 & -25 \end{bmatrix}$$

Next, we multiply the third row by $-1/25$:

$$\begin{bmatrix} 1 & 0 & 13 & 15 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now we add -13 times the third row to the first row to get

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

And then we add -5 times the last row to the second row to get

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This is equivalent to

$$\begin{aligned} x \qquad\qquad &= 2 \\ y \qquad &= -1 \\ z &= 1 \end{aligned}$$

This is of course the same solution as before.

In general. The goal is to reduce the matrix to a form with all zeros (except in the rightmost column), except 1s down the diagonal. Then you can read off the answer. The strategy is simple: Put a 1 in the entry in row one, column one.

Then use the first row to clean out the first column. That is, add multiples of the first row to the other rows, to put zeros in the rest of the first column. Next, put a 1 into the second row, second column entry. (Multiply the second row by something to do this, or if convenient, swap another row with the second row to put a 1 where you need it.) Then use the second row to clean out the second column. You then use the third row to clean the third column. Keep going in this way if your system is bigger.

But the tricky bit is when you encounter a dependent system, a system that turns out to have infinitely many solutions. What happens when you try to implement the solution strategy is that you end up with a row (or rows) of all zeros. Say you start out with

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & 5x_3 & = & 9 \\ x_1 & - & x_2 & - & x_3 & = & -3 \\ & & x_2 & + & 3x_3 & = & 6 \end{array}$$

So you write it in matrix form and get

$$\begin{bmatrix} 1 & 1 & 5 & 9 \\ 1 & -1 & -1 & -3 \\ 0 & 1 & 3 & 6 \end{bmatrix}$$

and you try to put this into reduced row-echelon form. But at a certain point, you end up with

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

You cannot proceed any further because you cannot put a 1 in the third row, third column spot. This is actually the system

$$\begin{array}{ccccccc} x & & & + & 2z & = & 3 \\ & & y & + & 3z & = & 6 \\ & & & & 0 & = & 0 \end{array}$$

Note the last equation is true, but carries no information (we already know zero equals zero). So this is really two equations in three unknowns. The unknowns are "under-determined," and can assume more than one choice of values. Mathematically, what's going on is the original system had the property that the last equation was actually the sum of some multiples of the first two equations. So it "depends" on the other equations, and the system is said to be "dependent." So what appeared to be a system of three equations in three unknowns was really equivalent to a system of two equations in three unknowns.

Anyway, here is what you would do with this system. Solve the first two equations for x and y respectively, in terms of z:

$$\begin{aligned} x &= 3 - 2z \\ y &= 6 - 3z \end{aligned}$$

Now we can let z be any number we wish, say t, and then get x and y in terms of the same number t:

$$\begin{aligned} x &= 3 - 2t \\ y &= 6 - 3t \\ z &= t \end{aligned}$$

We call the arbitrary number t a "parameter," and we have expressed the solution set of the equation in "parametric" form.

Example. Now we try:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & + & 4x_5 & = & -1 \\ -x_1 & - & 2x_2 & & & - & 5x_4 & - & x_5 & = & 3 \\ 2x_1 & + & 4x_2 & + & 2x_3 & + & 6x_4 & + & 9x_5 & = & 0 \end{array}$$

In matrix form:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 4 & -1 \\ -1 & -2 & 0 & -5 & -1 & 3 \\ 2 & 4 & 2 & 6 & 9 & 0 \end{bmatrix}$$

But this is how far we can reduce this:

$$\begin{bmatrix} 1 & 2 & 0 & 5 & 0 & -5 \\ 0 & 0 & 1 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(This is in reduced row-echelon form: every row begins with a 1, any column that contains one of these lead 1s has 0 in all remaining entries, and the lead 1s step to the right as you go down the matrix. See the book for a more detailed description.) Now we write this as a system of equations:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & & & + & 5x_4 & & & = & -5 \\ & & & & x_3 & - & 2x_4 & & & = & 4 \\ & & & & & & & & x_5 & = & 2 \end{array}$$

What we do now is solve for the "lead variables," the ones corresponding to the 1s in the reduced row-echelon form matrix, in terms of the remaining ("free") variables. We have

$$\begin{aligned} x_1 &= -5 - 2x_2 - 5x_4 \\ x_3 &= 4 + 2x_4 \\ x_5 &= 2 \end{aligned}$$

Now we let the free variables take any values we want: $x_2 = s$ and $x_3 = t$ say, and we can then write the entire solution set as:

$$\begin{aligned} x_1 &= -5 - 2s - 5t \\ x_2 &= s \\ x_3 &= 4 + 2t \\ x_4 &= t \\ x_5 &= 2 \end{aligned}$$

Example.

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & 5x_3 & = & 9 \\ x_1 & - & x_2 & - & x_3 & = & -3 \\ & & x_2 & + & 3x_3 & = & 7 \end{array}$$

or

$$\begin{bmatrix} 1 & 1 & 5 & 9 \\ 1 & -1 & -1 & -3 \\ 0 & 1 & 3 & 7 \end{bmatrix}$$

But the reduced row-echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which stands for the system

$$\begin{array}{ccccccc} x_1 & & & + & 5x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \\ & & & & 0 & = & 1 \end{array}$$

But the last equation, $0 = 1$, is of course false. So the last system has no solutions, and likewise, the first system has no solutions since the systems are equivalent. We call this system "inconsistent."

Technology. Row-reducing matrices is of course tedious to do by hand, and there's the additional problem that doing calculations by hand is prone to error. If you need to row-reduce matrices, or do other linear algebra work, certain calculators or software can do these operations for you. Texas Instruments TI-83 and TI-84 calculators, for example, will let you enter a matrix and will row-reduce the matrix for you.

A much more powerful tool is the free (open source) program Octave (also known as GNU/Octave). This is easy to find and download. It's a spartan program, with an old-fashioned command-line interface. But it isn't too difficult to use to perform relatively simple operations. For example, to define a matrix, one might enter:

```
> a = [1,3,5;2,4,6;7,7,7]
```

(The greater-than sign is the prompt where you enter the command.) It will display the matrix:

```
a =
   1  3  5
   2  4  6
   7  7  7
```

Then you enter

```
> rref(a)
```

and it will give you:

```
ans =
   1  0 -1
   0  1  2
   0  0  0
```

Another interesting resource is the web site `www.wolframalpha.com`. Try entering: `RowReduce[{{1,2,3},{4,5,6},{7,7,7}}]`. This same web site will perform many kinds of mathematical operations, and often give more information than just the answer. It also will solve problems from physics and other areas of science (it's part search engine and part computer algebra system).