Homework Assignment 6 Solutions

4. Let V be the set R^2 of real numbers, with the following operations: Let

$$(x,y)+(x',y')=\big(x+x',y+y'\big) \quad \text{and} \quad k(x,y)=(0,0).$$

So V has the usual vector addition, but a peculiar scalar product. Determine if V is a vector space with these operations. If it is not a vector space, list all axioms that fail to hold.

To determine if V is a vector space, we must test the axioms for a vector space. Here are the axioms for a vector space:

1. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in V, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.
4. There is a "zero vector" $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For all $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is in V then $k\mathbf{u}$ is in V.
7. For all vectors $\mathbf{u}$ in V and all scalars k and m we have $k(m\mathbf{u})=(km)\mathbf{u}$.
8. For all $\mathbf{u}$, $\mathbf{v}$ and k we have $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.
9. For all $\mathbf{u}$ and k and m, we have $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{v}$.
10. There is a scalar called 1 such that $1\mathbf{u}=\mathbf{u}$ for all vectors $\mathbf{u}$ in V.

Now the first four axioms will hold, because vector addition in this problem is exactly the same as ordinary vector addition for R^2.

Axiom 5 is a bit tricky. It turns out this axiom *does* hold in this vector space. Given a vector $\mathbf{u}$, we need a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$. If $\mathbf{u}=(x,y)$ then $-\mathbf{u}=(-x,-y)$. This works since $\mathbf{u}+(-\mathbf{u})=(x,y)+(-x,-y)=(x-x,y-y)=(0,0)=\mathbf{0}$. What's tricky here is that we would like to write $-\mathbf{u}=(-1)\mathbf{u}$. This works in actual vector spaces (Theorem 5.1.1c in Anton), but not in this fake vector space. Here, $(-1)\mathbf{u}=(-x,0)\neq(-x,-y)$. But this doesn't contradict Axiom 5, since Axiom 5 is satisfied by $-\mathbf{u}=(-x,-y)$.

Axiom 6 holds, since $k\mathbf{u}$ by definition is a vector in R^2, namely $(kx,0)$.

Now consider Axiom 7. This says $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$. Here, let $\mathbf{u}=(x,y)$ and $\mathbf{v}=(x',y')$. Then $k(\mathbf{u}+\mathbf{v})=k\big((x,y)+(x',y')\big)=k(x+x',y+y')=(k(x+x'),0)$ while $k\mathbf{u}+k\mathbf{v}=k(x,y)+k(x',y')=(kx,0)+(kx',0)=(kx+kx',0)=(k(x+x'),0)$. So we have verified that $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.

Axioms 7, 8 and 9 are similar to Axiom 7. They also hold.

Axiom 10 does not hold. If we let $\mathbf{u}=(x,y)$ then $1\mathbf{u}=1(x,y)=(1\cdot x,0)=(x,0)\neq(x,y)=\mathbf{u}$.

So in conclusion, all the axioms hold except 10, and this set is not a vector space with the given operations.

5. Let V be the set R^2 of real numbers, with the following operations: Let

$$(x,y)+(x',y')=(2x+2x',2y+2y') \quad \text{and} \quad k(x,y)=(kx,ky).$$

So V has the usual scalar product, but a peculiar vector addition. Determine if V is a vector space with these operations. If it is not a vector space, list all axioms that fail to hold.

For V, the scalar product is the ordinary scalar product on R^2. So Axioms 6, 7 and 10, which do not involve vector addition, will hold. Axiom 1 will also hold, because the sum of two vectors by definition is something in $V = R^2$.

Axiom 2 holds. To verify this, we write $\mathbf{u}=(x,y)$ and $\mathbf{v}=(x',y')$. Then $\mathbf{u}+\mathbf{v}=(x,y)+(x',y')=(2x+2x',2y+2y')=(2x'+2x,2y'+2y)=(x',y')+(x,y)=\mathbf{v}+\mathbf{u}$. Note that $2x'+2x=2x+2x'$ and $2y'+2y=2y+2y'$ because this is just ordinary arithmetic for real numbers.

But Axiom 3 does not hold. We are supposed to verify $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. We can let $\mathbf{u}=(x_1,y_1)$, $\mathbf{v}=(x_2,y_2)$ and $\mathbf{w}=(x_3,y_3)$. Then

$$\begin{aligned}\mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (x_1,y_1)+\big((x_2,y_2)+(x_3,y_3)\big)\\ &= (x_1,y_1)+(2x_2+2x_3,2y_2+2y_3)\\ &= \big(2x_1+2(2x_2+2x_3),\ 2y_1+2(2y_2+2y_3)\big)\\ &= (2x_1+4x_2+4x_3,\ 2y_1+4y_2+4y_3)\end{aligned}$$

but

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= ((x_1,y_1)+(x_2,y_2))+(x_3,y_3)\\ &= (2x_1+2x_2,2y_1+2y_2)+(x_3,y_3)\\ &= (2(2x_1+2x_2)+2x_3,\ 2(2y_1+2y_2)+2y_3)\\ &= (4x_1+4x_2+2x_3,\ 4y_1+4y_2+2y_3)\end{aligned}$$

so $\mathbf{u}+(\mathbf{v}+\mathbf{w})\neq(\mathbf{u}+\mathbf{v})+\mathbf{w}$ (unless certain components of these vectors are zero).

Axiom 4 also fails. Here, let $\mathbf{u}=(x,y)$. Then $\mathbf{u}+\mathbf{0}=(x,y)+(0,0)=(2x+2\cdot 0,2y+2\cdot 0)=(2x,2y)\neq\mathbf{u}$.

Axiom 5 does hold, if for $\mathbf{u}=(x,y)$ we let $-\mathbf{u}=(-x,-y)$; we do have $\mathbf{u}-\mathbf{u}=\mathbf{0}$.

And finally, Axioms 8 and 9 fail. Here's Axiom 9: If $\mathbf{u}=(x,y)$ then $(k+m)\mathbf{u}=(k+m)(x,y)=\big((k+m)x,(k+m)y\big)$ but $k\mathbf{u}+m\mathbf{u}=(kx,ky)+(mx+my)=(2kx+2mx,2ky+2my)$.

To summarize, this set is not a vector space with the given operations. Axioms 3, 4, 8 and 9 fail.

6. Let V be the set of all 2×2 matrices of the form $\begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix}$, given the usual matrix addition and scalar multiplication. Determine if this is a vector space. If it is not a vector space, list all axioms that fail to hold.

   Since the operations this set is given are the ordinary operations on M_{22}, all we need to consider are Axioms 1 and 6, closure under vector addition and closure under scalar multiplication.

   For addition, we consider two vectors of the given form, say $\begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix}$ and $\begin{bmatrix} a' & a'+b' \\ 0 & b' \end{bmatrix}$. Their sum must be of the same form. Their sum is

$$\begin{bmatrix} a+a' & (a+b)+(a'+b') \\ 0 & b+b' \end{bmatrix} = \begin{bmatrix} a+a' & (a+a')+(b+b') \\ 0 & b+b' \end{bmatrix}$$

   so we see that it *is* of the needed form. So Axiom 1 holds.

   Axiom 6 also holds. This time, we consider any scalar k and any matrix $\begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix}$. Then $k \begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & k(a+b) \\ k \cdot 0 & kb \end{bmatrix} = \begin{bmatrix} ka & ka+kb \\ 0 & kb \end{bmatrix}$, which is of the desired form.