Solutions for the Final Exam

1. Find the limits:

   (a) $\lim_{x\to 4} \frac{x^2-3x-4}{x^2-2x-8} = \lim_{x\to 4} \frac{(x-4)(x+1)}{(x-4)(x+2)} = \lim_{x\to 4} \frac{x+1}{x+2} = \frac{5}{6}.$

   (b) $\lim_{x\to\infty} \frac{1-x-x^2}{2x^2+5} = \lim_{x\to\infty} \frac{-x^2}{2x^2} = \lim_{x\to\infty} \frac{-1}{2} = -\frac{1}{2}$

2. Complete the definition of the limit $\lim_{t\to c} g(t) = L.$

   For every number $\epsilon > 0$ there is $\delta > 0$ such that if $0 < |t-c| < \delta$ then $|g(t) - L| < \epsilon.$

3. Use the Intermediate Value Theorem to show that there is a solution of the equation $x^3 + 2x = 7$ in the interval $(1, 2)$.

   Let $f(x) = x^3 + 2x$. Now $f(1) = 3$ and $f(2) = 12$. Since $f(1) < 7 < f(2)$, by the IVT there is a number c in the interval $(1, 2)$ such that $f(c) = 7$. So $c^3 + 2c = 7$, so the equation has a solution in the interval.

4. Use the definition of derivative to verify that if $f(x) = x^2+3x+4$ then $f'(x) = 2x+3$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2+3(x+h)+4)-(x^2+3x+4)}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2+3x+3h+4-(x^2+3x+4)}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2+3h}{h} = \lim_{h\to 0} \frac{(2x+h+3)h}{h} \\
&= \lim_{h\to 0} (2x+h+3) = 2x+3
\end{aligned}
$$

5. Find the derivative of each function.

   (a) $y = \frac{x^4}{x^3+2}$

   $\frac{dy}{dx} = \frac{4x^3(x^3+2) - x^4(3x^2)}{(x^3+2)^2} = \frac{x^3(x^3+8)}{(x^3+2)^2}$ using the quotient rule.

   (b) $f(x) = x^2 \tan x$

$f'(x) = 2x \tan x + x^2 \sec^2 x$ using the product rule.

6. Find the derivative of each function.

   (a) $F(x) = \sqrt[4]{1+x^3}$

   $F(x) = (1+x^3)^{1/4}$ so $F'(x) = \frac{1}{4}(1+x^3)^{-3/4}(3x^2) = \frac{3}{4}x^2(1+x^3)^{-3/4}$ using the chain rule.

   (b) $y = \left(\dfrac{x^2+1}{x^2-1}\right)^3$

   This is $y = u^3$ where $u = \dfrac{x^2+1}{x^2-1}$. Now $\dfrac{dy}{du} = 3u^2$ and $\dfrac{du}{dx} = \dfrac{-4x}{(x^2-1)^2}$ (using the quotient rule), so $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = 3u^2\left(\dfrac{-4x}{(x^2-1)^2}\right) = -\dfrac{12x(x^2+1)^2}{(x^2-1)^4}$

7. Find dy/dx by implicit differentiation: $x^2 + xy - y^2 = 3$

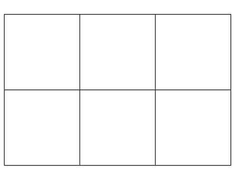
   This is: $2x + y + xy' - 2yy' = 0$ or $2x + y = -xy' + 2yy'$ so $2x + y = (-x + 2y)y'$. We get $y' = \dfrac{2x+y}{-x+2y}$

8. Find the differential dy and evaluate dy for the given values of x and dx.

   $y = x^3 + 4x, \quad x = 2, \quad dx = -0.05.$

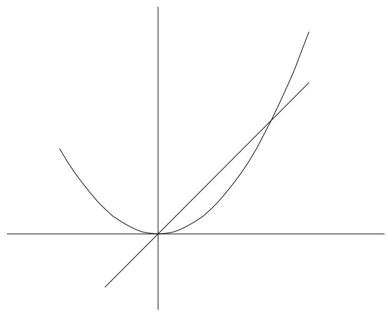
   This is $dy = f'(x)\,dx = (3x^2+4)\,dx = \big(3(2^2)+4\big)(-0.05) = -0.8.$

9. A farmer needs to make 6 rectangular pens as shown, using 24 yards of fence. What dimensions result in the greatest possible area inside the pens?



   If the six pens (taken together) are x wide and y tall, then the amount of fence used is $3x + 4y = 24$. The area of the six pens taken together is $A = xy$. Solve the first equation for y to get $y = \frac{1}{4}(24-3x)$, so $A = xy = \frac{1}{4}x(24-3x) = 6x - \frac{3}{4}x^2$. Then $dA/dx = 6 - \frac{3}{2}x$. Set that equal to zero and solve to find the critical point: $6 - \frac{3}{2}x = 0$ or $x = 4$. Then $y = \frac{1}{4}(24-3x) = \frac{1}{4}(24-3(4)) = 3$. So the pens taken together are 4×3 yards; each individual pen is $\frac{4}{3} \times \frac{3}{2}$ yards (or 4×4.5 feet).

10. Find the area between the two curve $y = 6x$ and $y = x^2$.



We need to find where the curves intersect, so solve $y = 6x$ and $y = x^2$ together to get $6x = x^2$, which leads to $x = 0, 6$. So the area is $A = \int_a^b (f(x) - g(x))\, dx = \int_0^6 (6x - x^2)\, dx = 36$.

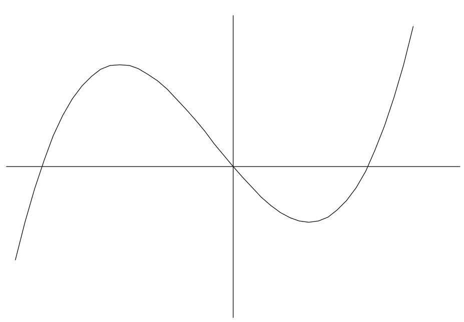
11. Find the volume of the solid obtained by rotating the region bounded by the curves $y = 1 + 2x, \quad y = 0, \quad x = 1, \quad x = 2$ about the x-axis.

We integrate over x to obtain $V = \int_a^b \pi \left(f(x)\right)^2\, dx = \int_1^2 \pi(1 + 2x)^2\, dx = 49\pi/3$.

12. Find the absolute maximum and absolute minimum values of $f(x) = 3x^2 - 12x + 5$ on the interval $[0, 3]$.

First, find any critical points in the interval: $f'(x) = 6x - 12$, so $6x - 12 = 0$ gives $x = 2$. Now just compare the values of $f(x)$ at $x = 2$ as well as at the endpoints $x = 0$ and $x = 3$. We get $f(0) = 5$, $f(2) = -7$ and $f(3) = -4$. So the maximum value is $f(0) = 5$ and the minimum value is $f(2) = -7$.

13. Find the intervals of increase and decrease, and the local minimums and maximums, of the function $f(x) = 2x^3 + 3x^2 - 36x$.



This involves the first derivative: $f'(x) = 6x^2 + 6x - 36 = 6(x - 2)(x + 3)$. So the critical points are $x = 2$ and $x = -3$. So the three intervals of interest are $x < -3$, $-3 < x < 2$ and $2 < x$. We might test these intervals by putting numbers in them into $f'(x)$. But from the graph provided, it is clear that the function is increasing on $x < -3$ and on $2 < x$, and decreasing on $-3 < x < 2$. This in turn tells us $x = -3$ is a local maximum and $x = 2$ is a local minimum.

14. Find the intervals of concavity and all inflection points of the function $f(x) = 2x^3 + 3x^2 - 36x$.

    This involves the second derivative: $f''(x) = 12x + 6$. Now the function is concave up when $12x + 6 > 0$ or $x > -\frac{1}{2}$. It is concave down when $x < -\frac{1}{2}$, and $x = -\frac{1}{2}$ is an inflection point.

15. Find the function f such that $f''(x) = 6x - 4$, $f(1) = 8$ and $f'(1) = 5$.

    First, $f'(x) = 3x^2 - 4x + C$. Since $f'(1) = 5$, we know $3(1^2) - 4(1) + C = 5$ so $C = 6$. That is, $f'(x) = 3x^2 - 4x + 6$.

    Now $f(x) = x^3 - 2x^2 + 6x + K$. Since $f(1) = 8$, we have $1^3 - 2(1^2) + 6(1) + K = 8$ or $K = 3$. So $f(x) = x^3 - 2x^2 + 6x + 3$.

16. Evaluate the integral $\displaystyle\int_1^4 (3t^2 + 4t + 5)\, dt$

    Using the Fundamental Theorem of Calculus, $\displaystyle\int_1^4 (3t^2+4t+5)\, dt = \left[t^3 + 2t^2 + 5t\right]_1^4 =$ $(64 + 3(16) + 5(4)) - (1 + 2 + 5) = 108$.

17. Evaluate the integral $\displaystyle\int x^2(x^3 + 4)^5\, dx$

    Use the substitution $u = x^3 + 4$ and $du = 3x^2\, dx$ or $\frac{1}{3}\, du = x^2\, dx$. So $\displaystyle\int x^2(x^3 + 4)^5\, dx = \frac{1}{3}\int u^5\, du = \frac{1}{18}u^6 + C = \frac{1}{18}\left(x^3 + 4\right)^6 + C$.

18. Find the inverse function of $f(x) = x^3 - 4$.

    Solve $x = y^3 - 4$ to obtain $y = \sqrt[3]{x + 4}$. So $f^{-1}(x) = \sqrt[3]{x + 4}$.

19. Find each derivative.

    (a) $f(x) = x^3 \ln x$

    $f'(x) = 3x^2 \ln x + x^3\left(\frac{1}{x}\right) = 3x^2 \ln x + x^2$, using the product rule.

    (b) $y = e^{-x^2}$

    $y' = e^{-x^2}(-2x) = -2xe^{-x^2}$, using the chain rule.

20. Find each integral.

    (a) $\displaystyle\int \frac{x}{x^2 + 1}\, dx$

Let $u = x^2 + 1$ so $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. So $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du =$ $\frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$.

(b) $\displaystyle\int_0^4 e^{-2x}\,dx$

Recall $\displaystyle\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$, so $\displaystyle\int_0^4 e^{-2x}\,dx = -\left.\frac{e^{-2x}}{2}\right]_0^4 = \frac{1}{2} - \frac{1}{2}e^{-8}$