Test 1 Solutions

1. Evaluate the infinite limits.

   (a) $\lim_{x\to 2^-} \frac{1}{x^2-4}$

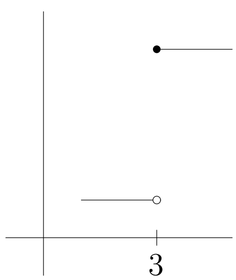
   (b) $\lim_{x\to 2^+} \frac{1}{x^2-4}$

   For (a), in the limit, x is close to 2 but $x < 2$, so use $x = 1.999$ or a similar number. We get $1/(x^2-4) = 1/(1.999^2-4) = -250.06$. So this limit is $-\infty$. For (b), $x > 2$, so use $x = 2.001$. This time, $1/(x^2-4) = 1/(2.001^2-4) = 249.9$, so the second limit is ∞.

2. Sketch the graph of a function f with the following properties:

   $$\lim_{x\to 3^-} f(x) = 1$$
   $$\lim_{x\to 3^+} f(x) = 5$$
   $$f(3) = 5$$

   This may look something like:



3. Find the limit $\lim_{x\to 4} \frac{x^2-7x+12}{x^3-16x}$

   $$\lim_{x\to 4} \frac{x^2-7x+12}{x^3-16x} = \lim_{x\to 4} \frac{(x-3)(x-4)}{x(x-4)(x+4)} = \lim_{x\to 4} \frac{x-3}{x(x+4)} = \frac{4-3}{4(4+4)} = \frac{1}{32}.$$

4. Find the limit $\lim_{x\to 9} \frac{\sqrt{x+7}-4}{x-9}$

   $$\lim_{x\to 9} \frac{\sqrt{x+7}-4}{x-9} = \lim_{x\to 9} \frac{(\sqrt{x+7}-4)(\sqrt{x+7}+4)}{(x-9)(\sqrt{x+7}+4)} = \lim_{x\to 9} \frac{(x+7)-16}{(x-9)(\sqrt{x+7}+4)}$$
   $$= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x+7}+4)} = \lim_{x\to 9} \frac{1}{\sqrt{x+7}+4} = \lim_{x\to 9} \frac{1}{\sqrt{9+7}+4} = \frac{1}{\sqrt{16}+4} = \frac{1}{8}.$$

5. State the definition of the limit $\lim_{x\to c} f(x) = M$.

For every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - M| < \epsilon$.

6. Consider the limit $\lim_{x\to 4}(5x - 8) = 12$.

   (a) How close must x be to 4, for $5x - 8$ to be within 0.01 units of 12?

   (b) How close must x be to 4, for $5x - 8$ to be within ϵ units of 12?

For $5x - 8$ to be within 0.01 units of 12, we must have $|5x - 8 - 12| < 0.01$. This is $|5x - 20| < 0.01$ or $5|x - 4| < 0.01$. So $|x - 4| < 0.002$. That is, x must be within 0.002 units of 4.

For part (b), we solve $|5x - 8 - 12| < \epsilon$ to get $|x - 4| < \epsilon/5$. So x must be within $\epsilon/5$ units of 4. (By the way, in terms of the definition of limit, we have $\delta = \epsilon/5$ in this problem.)

7. Indicate why each function is discontinuous at $x = 2$.

   (a) $f(x) = \dfrac{x^2 - 4}{x - 2}$

   (b) $f(x) = \dfrac{x^2 + 4}{x - 2}$

For part (a), the discontinuity is removable since $\lim_{x\to 2} f(x)$ exists. This is $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{x^2 - 4}{x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x+2)}{x - 2} = \lim_{x\to 2}(x + 2) = 4$.

But for part (b), the discontinuity is infinite since this time, $\lim_{x\to 2} \dfrac{x^2 + 4}{x - 2} = \pm\infty$ (the function doesn't factor).

8. Find c such that the following function is continuous: $f(x) = \begin{cases} x^2 & \text{if } x \leq 3 \\ cx - 6 & \text{if } x > 3 \end{cases}$

For continuity, we must have $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$, which is $\lim_{x\to 3^-} x^2 = \lim_{x\to 3^+} (cx - 6)$. But this is $9 = 3c - 6$, which solves to give $c = 5$.

9. Use the Intermediate Value Theorem to show that the given equation has a solution in the given interval: $x^3 + 2x = 4, \quad (1, 2)$

Let $f(x) = x^3 + 2x - 4$. Then $f(1) = -1$ but $f(2) = 8$. So $f(1) < 0 < f(2)$. So by the IVT, there is a number c in the interval $(1, 2)$ such that $f(c) = 0$. That is, $c^3 + 2c - 4 = 0$, so $c^3 + 2c = 4$. So c is a solution to the equation in the interval $(1, 2)$.

10. Find the slope of the curve $y = x^2 + 2x$ at $x = 3$ by setting up and evaluating an appropriate limit.

$m = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim_{x \to 3} \dfrac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \dfrac{x^2 + 2x - 15}{x - 3} = \lim_{x \to 3} \dfrac{(x-3)(x+5)}{x - 3} = \lim_{x \to 3}(x + 5) = 8.$

11. A rock falls off of a 100 ft cliff. It is $y = 100 - 16t^2$ ft above the ground t seconds later. Find its velocity 2 seconds after it falls.

$v = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \dfrac{100 - 16(2+h)^2 - 36}{h} = \lim_{h \to 0} \dfrac{100 - 16(4 + 4h + h^2) - 36}{h} = \lim_{h \to 0} \dfrac{-64h - 16h^2}{h} = \lim_{h \to 0} \dfrac{(-64 - 16h)h}{h}$
$= \lim_{h \to 0}(-64 - 16h) = 64.$

Note, in problem 10, I set up the limit as $x \to a$ but in problem 11, I set up the limit as $h \to 0$. I could have just as easily done either problem the other way; the calculations would have been similar, and of course, completely correct either way.