(4.4) Limits at infinity and asymptotes

The main topic here are limits of the sort $\lim_{x\to\infty} f(x)$. This means, roughly, as x gets bigger and bigger (beyond any bounds), then what does $f(x)$ get close to?

So we say $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to L as desired by making x sufficiently large.

Here's the technical definition: $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$, there is a number N such that $|f(x) - L| < \epsilon$ whenever $x > N$.

Example. $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

Example. $\lim_{x\to\infty} x^2 = \infty$. (I.e., as x gets bigger and bigger, so does x^2. Note the technical definition of $\lim_{x\to\infty} f(x) = \infty$ at the end of the section in the book.)

Example. $\lim_{x\to\infty} \frac{x^3+5x}{2x^3+8}$

We can work this as follows:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3+5x}{2x^3+8} &= \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{5x}{x^3}}{\frac{2x^3}{x^3}+\frac{8}{x^3}} \\ &= \lim_{x\to\infty} \frac{1+\frac{5}{x^2}}{2+\frac{8}{x^3}} \\ &= \frac{1+0}{2+0} = \frac{1}{2}\end{aligned}$$

Example. $\lim_{x\to\infty} \frac{x^4+5x}{2x^3+8} = \lim_{x\to\infty} \frac{\frac{x^4}{x^4}+\frac{5x}{x^4}}{\frac{2x^3}{x^4}+\frac{8}{x^4}} = \lim_{x\to\infty} \frac{1+\frac{5}{x^3}}{\frac{2}{x}+\frac{8}{x^4}} = \frac{1+0}{0+0}$ does not exist. (Actually, we may say that the limit is ∞.)

Example. $\lim_{x\to\infty} \frac{x^2+5x}{2x^3+8}$ We can do this in a simpler manner. Drop all but the highest-power term in the numerator and the highest-power term in the denominator:
$\lim_{x\to\infty} \frac{x^2+5x}{2x^3+8} = \lim_{x\to\infty} \frac{x^2}{2x^3} = \lim_{x\to\infty} \frac{1}{2x} = 0$.

Example. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2+1}} = \sqrt{\lim_{x\to\infty} \frac{x^2}{x^2+1}} = \sqrt{1} = 1$

Note $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$. (Here, we use the fact that the square root function is continuous, so if $\lim f(x) = L$ it follows that $\lim \sqrt{f(x)} = \sqrt{L}$.)

Example. Why are limits at infinity interesting? Suppose you fall out of an airplane. Your velocity increases, but only to a certain amount (the so-called "terminal velocity"). Suppose your velocity t seconds after falling out of the plane is $v(t)$. Then $\lim_{t\to\infty} v(t) = v_\infty$ (some constant value). (Of course, t might not get to be very large.)

Example. It can be shown that

$$\lim_{n\to\infty} \frac{n^n e^{-n}\sqrt{2\pi n}}{n!} = 1.$$

So $n! \approx n^n e^{-n}\sqrt{2\pi n}$ for large n. This is called Stirling's formula. For example, if $n = 10$, we have $10! \approx 10^{10} e^{-10}\sqrt{20\pi} = 3{,}498{,}695.6$ ($10! = 3{,}628{,}800$.) For $n = 100$, the formula gives $100! \approx 9.325 \times 10^{157}$ but $100! = 9.333 \times 10^{157}$. This is an "asymptotic" approximation. Note the absolute error is about 7.77×10^{154} (really, really big!) but the relative error is about 0.08%.

Horizontal asymptotes

We say $y = L$ is a horizontal asymptote for $y = f(x)$ if $\lim_{x\to\infty} f(x) = L$ or if $\lim_{x\to-\infty} f(x) = L$.

Example. $y = \frac{1}{x-3}$. Here $L = 0$ so the horizontal asymptote is $y = 0$. Note there's a vertical asymptote too.

Example. $y = \frac{x^2}{1+x^2}$. Since $\lim_{x\to\pm\infty} \frac{x^2}{1+x^2} = 1$, we know $y = 1$ is the horizontal asymptote. We note $y' = \frac{2x}{(1+x^2)^2}$ and $y'' = \frac{2-6x^2}{(1+x^2)^3}$, so it isn't hard to graph the function (using increase/decrease and concavity). (There is an absolute minimum at $x = 0$ and there are inflection points at $x = \pm\sqrt{1/3}$, $y = 1/4$, circled on the graph below.)