## (3.3) Rules for Differentiation

We would like to avoid the limit calculations we used to find $f'$ for functions such as $f(x) = \frac{2x}{x-3}$. We will collect simple rules that enable us to differentiate complicated functions without direct computation of the limit.

Recall we've seen derivatives such as $\frac{d}{dx}\big(x^2\big) = 2x$ and $\frac{d}{dx}\big(x^3\big) = 3x^2$. By setting up and computing the appropriate limits, we can also show $\frac{d}{dx}\big(x^4\big) = 4x^3$ and $\frac{d}{dx}\big(x^5\big) = 5x^4$. A pattern becomes obvious, a pattern that can be proved to always work. This is the **power rule**:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

In particular, $\frac{d}{dx}c = 0$ for constants $c$ and $\frac{d}{dx}x = 1$. It's actually true for any real number $n$, including negative numbers and fractions.

**Examples.**

1. $f(x) = \frac{1}{x^7}$.

   This is $f(x) = x^{-7}$ so $f'(x) = -7x^{-8} = -\frac{7}{x^8}$. (Note, a common mistake is to write $-7x^{-6}$ as the answer, but if you subtract 1 from $-7$, you get $-8$.)

2. $y = \sqrt[3]{x}$.

   This is $y = x^{1/3}$ so $\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$. Here, $\frac{1}{3} - 1 = \frac{1}{3} - \frac{3}{3} = -\frac{2}{3}$. Now if we wanted to write the answer with a root, we would write $\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$.

The next three rules deal with constant multiples, sums and differences of functions:

$$\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$$

$$\frac{d}{dx}\Big(f(x) + g(x)\Big) = f'(x) + g'(x)$$

$$\frac{d}{dx}\Big(f(x) - g(x)\Big) = f'(x) - g'(x)$$

So we can take derivatives of **polynomials**:

$$\frac{d}{dx}\left(3x^5 - \tfrac{7}{2}x^4 - 5x + 7\right) = 3(5x^4) - \tfrac{7}{2}(4x^3) - 5 + 0 = 15x^4 - 14x^3 - 5.$$

We refer to this as "taking derivatives term-by-term."

For more complicated functions:

**Product Rule:**

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x)$$

**Quotient Rule:**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

**Examples.**

1. $f(x) = (x^2)(x^3)$

   $f'(x) = (2x)(x^3) + (x^2)(3x^2) = 5x^4$.

   Note, if you didn't use the product rule, and just simply wrote $f'(x) = (2x)(3x^2) = 6x^3$, you get a wrong answer.

2. $y = (x^3 + 5x + 2)(x^2 + 7)$

   $$\frac{dy}{dx} = (3x^2 + 5)(x^2 + 7) + (x^3 + 5x + 2)(2x).$$

3. $f(x) = \sqrt[3]{x}(1 + x^3)$

   $$f'(x) = \left(\frac{1}{3\sqrt[3]{x^2}}\right)(1 + x^3) + \sqrt[3]{x}(3x^2).$$

   But there's an easier way of doing this problem: $f(x) = x^{1/3}(1 + x^3) = x^{1/3} + x^{10/3}$ so $f'(x) = \frac{1}{3}x^{-2/3} + \frac{10}{3}x^{7/3}$.

4. $f(t) = \dfrac{t}{1 + t^2}$

   $$f'(t) = \frac{(1)(1 + t^2) - t(2t)}{(1 + t^2)^2} = \frac{1 - t^2}{(1 + t^2)^2}$$

5. $y = \dfrac{1-u}{1+u}$

   $\dfrac{dy}{du} = \dfrac{-2}{(1+u)^2}$

6. $f(t) = \dfrac{5}{\sqrt{t}}$

   $f'(t) = \dfrac{-5}{2t\sqrt{t}}$

   (The quotient rule isn't needed: $f(t) = 5t^{-1/2}$ so $f'(t) = -\frac{5}{2}t^{-3/2}$.)

7. $y = (1+x^2)^3$

   Use the product rule twice.

   Note $\dfrac{d}{dx}(1+x^2)^2 = \dfrac{d}{dx}\big((1+x^2)(1+x^2)\big) = 2x(1+x^2)+(1+x^2)(2x) = 4x(1+x^2)$. So

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1+x^2)^3 = \frac{d}{dx}\big((1+x^2)(1+x^2)^2\big)\\ &= (2x)(1+x^2)^2 + (1+x^2)\big(4x(1+x^2)\big) = 6x(1+x^2)^2.\end{aligned}$$

   (Later, we will learn the chain rule, which will give us an easier way of doing this problem.)

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**Why do these rules work?** Let's look at the product rule. This can be proved by setting up the appropriate limit:

$$\begin{aligned}\frac{d}{dx}\big(f(x)g(x)\big) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{\big(f(x+h)-f(x)\big)g(x+h)+f(x)\big(g(x+h)-g(x)\big)}{h}\\ &= \lim_{h\to 0}\left[\left(\frac{f(x+h)-f(x)}{h}\right)g(x+h)+f(x)\left(\frac{g(x+h)-g(x)}{h}\right)\right]\\ &= \left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)g(x)+f(x)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\\ &= f'(x)g(x)+f(x)g'(x).\end{aligned}$$

Note that we used the basic theorems about limits; for example, if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to b} g(x) = M$ then $\lim_{x\to a} f(x)g(x) = LM$. (This was to get from line four to line five.) We also used the fact that $\lim_{h\to 0} g(x+h) = g(x)$, which is true because $g$ is continuous at $x$, which in turn is true because of our assumption that $g$ is differentiable at $x$.